

Variational Methods for Computer Vision: Solution Sheet 2

Exercise: 07 November 2013

Part I: Theory

1.

$$\begin{aligned}(f(x, y) * k_1(x)) * k_2(y) &= \int \left(\int k_1(k) f(x - k, y - l) dk \right) k_2(l) dl \\&= \int \int f(x - k, y - l) k_1(k) k_2(l) dk dl \\&= \int \int f(x - k, y - l) \frac{1}{2\pi\sigma^2} e^{-\frac{k^2+l^2}{2\sigma^2}} dk dl \\&= \int \int f(x - k, y - l) K(k, l) dk dl \\&= f(x, y) * K(x, y)\end{aligned}$$

2. (a)

$$\begin{aligned}\nabla \tilde{f}(x) &= \nabla(f \circ R)(x) \\&= (\mathbf{D}_{Rx} f \circ \mathbf{D}_x R)^T \\&= (\mathbf{D}_{Rx} f \circ R)^T \\&= R^T \mathbf{D}_{Rx} f^T \\&= R^T \nabla f(Rx)\end{aligned}\tag{1}$$

Thus:

$$\begin{aligned}R \nabla \tilde{f}(x) &= RR^T \nabla f(Rx) \\&= \nabla f(Rx)\end{aligned}$$

(b)

$$\begin{aligned}|\nabla \tilde{f}(x)| &\stackrel{(1)}{=} |R^T \nabla f(Rx)| \\&= \sqrt{(R^T \nabla f(Rx))^T (R^T \nabla f(Rx))} \\&= \sqrt{\nabla f(Rx)^T R R^T \nabla f(Rx)} \\&= \sqrt{\nabla f^T \nabla f} \\&= |\nabla f(Rx)|\end{aligned}$$

(c)

$$\begin{aligned}\Delta \tilde{f}(x) &= \operatorname{div}(\underbrace{\nabla \tilde{f}(x)}_{R^T \nabla f(Rx)}) \\&= \operatorname{div} \begin{pmatrix} \cos(\alpha) f_x + \sin(\alpha) f_y \\ -\sin(\alpha) f_x + \cos(\alpha) f_y \end{pmatrix} = \tilde{f}_{xx} + \tilde{f}_{yy}\end{aligned}$$

Thus:

$$\begin{aligned}
\tilde{f}_{xx} &= \partial_x (\cos(\alpha)f_x + \sin(\alpha)f_y) \\
&= \partial_x \cos(\alpha)f_x + \partial_x \sin(\alpha)f_y \\
&= \cos^2(\alpha)f_{xx} + \cos(\alpha)\sin(\alpha)f_{xy} + \cos(\alpha)\sin(\alpha)f_{yx} + \sin^2(\alpha)f_{yy}
\end{aligned}$$

And:

$$\begin{aligned}
\tilde{f}_{yy} &= \partial_y (-\sin(\alpha)f_x + \cos(\alpha)f_y) \\
&= \sin^2(\alpha)f_{xx} - \cos(\alpha)\sin(\alpha)f_{xy} - \cos(\alpha)\sin(\alpha)f_{yx} + \cos^2(\alpha)f_{yy}
\end{aligned}$$

Therefore:

$$\begin{aligned}
\tilde{f}_{xx} + \tilde{f}_{yy} &= f_{yy}(\underbrace{\sin^2(\alpha) + \cos^2(\alpha)}_{=1}) + f_{xx}(\underbrace{\sin^2(\alpha) + \cos^2(\alpha)}_{=1}) \\
&= \Delta(f(Rx))
\end{aligned}$$

3. (a)

$$\begin{aligned}
\operatorname{div}(g\nabla u(x)) &= \frac{\partial}{\partial x}g\frac{\partial}{\partial x}u(x) + \frac{\partial}{\partial y}g\frac{\partial}{\partial y}u(x) \\
&= g\Delta u(x)
\end{aligned}$$

(b)

$$\begin{aligned}
\operatorname{div}(g(x)\nabla u(x)) &= \frac{\partial}{\partial x}\left(g(x)\frac{\partial}{\partial x}u(x)\right) + \frac{\partial}{\partial y}\left(g(x)\frac{\partial}{\partial y}u(x)\right) \\
&= \frac{\partial^2}{\partial x^2}u(x)g(x) + \frac{\partial^2}{\partial y^2}u(x)g(x) + \frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}u(x) + \frac{\partial}{\partial y}g(x)\frac{\partial}{\partial y}u(x) \\
&= g(x)\Delta u(x) + \langle \nabla g(x), \nabla u(x) \rangle
\end{aligned}$$