Variational Methods for Computer Vision: Solution Sheet 3

Exercise: 14 November 2013

Part I: Theory

1. (a) Suppose x^* is a local but not a global minimizer. Then we can find a point $z \in \mathbb{R}^n$ with $f(z) < f(x^*)$. Consider the line segment joining z and x^*, that is defined as follows:

$$x = \lambda z + (1-\lambda)x^* \text{ for some } \lambda \in (0,1)$$

Then by convexity:

$$f_\lambda(x) \leq \lambda f(z) + (1-\lambda)f(x^*) < f(x^*)$$

This means that the neighborhood of x^* (any $\lambda \in [0,1)$) contains a point $f_\lambda(x)$ that has a lower energy which is in contradiction to the assumption.

(b) Reminder: The directional derivative of functional f in direction p is defined as

$$D(f(x),p) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon p) - f(x)}{\epsilon} = \nabla f(x)^T p$$

Then:

$$\begin{aligned}
\nabla f(x^*)^T(z - x^*) &= \lim_{\epsilon \to 0} \frac{f(x^* + \epsilon(z - x^*)) - f(x^*)}{\epsilon} \\
&\overset{\text{Convexity}}{\leq} \lim_{\epsilon \to 0} \frac{\epsilon f(z) + (1-\epsilon)f(x^*) - f(x^*)}{\epsilon} \\
&= f(z) - f(x^*) < 0
\end{aligned}$$

$\Rightarrow \nabla f(x^*)^T \neq 0 \Rightarrow x^*$ not stationary.

2. ”⇒”

Let f be convex and $(u,a),(v,b) \in \text{epi}f$. Then

$$\begin{aligned}
f(\lambda u + (1-\lambda)v &\leq \lambda f(u) + (1-\lambda)f(v) \\
&\leq \lambda a + (1-\lambda)b
\end{aligned}$$

Thus $\lambda(u,a) + (1-\lambda)(v,b) \in \text{epi}f$.

”⇐”

Let epi$f$ be convex. Define $a := f(x), b := f(y)$. Then $\begin{pmatrix} x \\ a \end{pmatrix}, \begin{pmatrix} y \\ b \end{pmatrix} \in \text{epi}f$. Because of the convexity of epi$f$, $\lambda \begin{pmatrix} x \\ a \end{pmatrix} + (1-\lambda)\begin{pmatrix} y \\ b \end{pmatrix} = \begin{pmatrix} \lambda x + (1-\lambda)y \\ \lambda a + (1-\lambda)b \end{pmatrix} \in \text{epi}f$.

Thus:

$$\begin{aligned}
f(\lambda x + (1-\lambda)y) &\leq \lambda a + (1-\lambda)b \\
&= \lambda f(x) + (1-\lambda)f(y) \\
&\Rightarrow f \text{ convex.}
\end{aligned}$$

3. (a)

$$\begin{aligned}\alpha f(\lambda x+(1-\lambda)y)+\beta g(\lambda x+(1-\lambda)y) &\le \alpha(\lambda f(x)+(1-\lambda)f(y))+\beta(\lambda g(x)+(1-\lambda)g(y))\\ &= \lambda(\alpha f(x)+\beta g(x))+(1-\lambda)(\alpha f(y)+\beta g(y))\end{aligned}$$

(b)

$$\begin{aligned}h(\lambda x+(1-\lambda)y) &= \max(f(\lambda x+(1-\lambda)y), g(\lambda x+(1-\lambda)y))\\ &\le \max(\lambda f(x)+(1-\lambda)f(y), \lambda g(x)+(1-\lambda)g(y))\\ &\le \lambda\max(g(x), f(x))+(1-\lambda)\max(f(y), g(y))\end{aligned}$$

(c) Counterexample: convince yourself that the graph $\min((x-1)^2, (x+1)^2)$ is not convex.

4.

$$\begin{aligned}h^{''} &= f(g(x))^{''}\\ &= (f^{'}(g(x)g^{'}(x))^{'}\\ &= \underbrace{f^{''}(g(x))}_{\ge 0}\underbrace{(g^{'}(x))^2}_{\ge 0}+f^{'}(g(x))\underbrace{g^{''}(x)}_{\ge 0}\end{aligned}$$

Thus:

$$h^{''}\ge 0 \Leftrightarrow f^{'}(g(x))\ge 0 \Leftrightarrow f(g(x)) \text{ monotonously increasing.}$$