Variational Methods for Computer Vision: Solution Sheet 5

Exercise: 28 November 2013

Part I: Theory

1. From exercise 1 of sheet 4 we know that the Euler-Lagrange equation of functionals involving

$$\mathcal{L}(u(x), u'(x), u''(x)) \quad \text{is} \quad \frac{\partial \mathcal{L}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial u'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial \mathcal{L}}{\partial u''}$$

   Using this result we get:

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial \mathcal{L}}{\partial C_s} + \frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{\partial \mathcal{L}}{\partial C_{ss}} &= -\frac{\mathrm{d}}{\mathrm{d}s}\left[\alpha(s)C'(s)\right] + \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left[\beta(s)C''(s)\right] \\
&= -\left[\alpha'(s)C'(s) + \alpha(s)C''(s)\right] + \frac{\mathrm{d}}{\mathrm{d}s}\left[\beta'(s)C''(s) + \beta(s)C'''(s)\right] \\
&= -\alpha'(s)C'(s) + \alpha(s)C''(s) + \beta''(s)C''(s) \\
&\quad + \beta'(s)C'''(s) + \beta'(s)C'''(s) + \beta(s)C''''(s)
\end{aligned}$$

2. (a)

$$L(C) := \int_0^1 |\dot{C}(s)|\mathrm{d}s = \int_0^1 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}$$

   So:

$$\begin{aligned}
\frac{\partial L}{\partial C'} &= \frac{\partial \sqrt{C'(s)^2}}{\partial C'(s)} \\
&= 2C'(s)\frac{1}{2\sqrt{C'(s)^2}} \\
&= \frac{C'(s)}{|C'(s)|}
\end{aligned}$$

   And:

$$\frac{\partial L}{\partial C} = -\mathrm{d}s\frac{C'(s)}{|C'(s)|}$$

(b)

$$\int_0^1 |C(m(s))'|\mathrm{d}s = \int_0^1 \left|\frac{\mathrm{d}C}{\mathrm{d}m}\frac{\mathrm{d}m}{\mathrm{d}s}\right|\mathrm{d}s$$

$$= \int_0^1 \left|\frac{\mathrm{d}C}{\mathrm{d}m}\right|\left|\frac{\mathrm{d}m}{\mathrm{d}s}\right|\mathrm{d}s$$

$$= \int_0^1 \left|\frac{\mathrm{d}C}{\mathrm{d}m}\right|\frac{\mathrm{d}m}{\mathrm{d}s}\mathrm{d}s$$

$$= \int_0^1 \left|\frac{\mathrm{d}C}{\mathrm{d}x}\right|\mathrm{d}x \qquad (*)$$

* Using integration by substitution with $x = m(s)$ and $\mathrm{d}x = m'(s)\mathrm{d}s$