Variational Methods for Computer Vision: Solution Sheet 8

Exercise: 10 January 2014

Part I: Theory

1. Using the property that the delta distribution $\delta(\phi)$ is the derivative of H, we get:

$$|\nabla H(\phi)| = |\delta(\phi) \cdot \nabla\phi(x)| = \delta(\phi)|\nabla\phi(x)|$$

So we can rewrite the functional in the following form:

$$\begin{aligned}\mathcal{L} &= f_1 \cdot H(\phi) + f_2(1 - H(\phi)) + \nu|\nabla H(\phi)| \\ &= (f_1 - f_2)H(\phi) + \nu\delta(\phi)|\nabla\phi(x)|\end{aligned}$$

So:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\nabla\phi} &= \frac{\partial}{\partial\nabla\phi}\nu|\nabla\phi|\delta(\phi) \\ &= \nu \cdot \delta(\phi) \cdot \frac{\nabla\phi}{|\nabla\phi|}\end{aligned}$$

And:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi} &= H'(\phi) \cdot (f_1 - f_2) + \nu\delta'(\phi)|\nabla\phi(x)| \\ &= \delta(\phi) \cdot (f_1 - f_2) + \nu\delta'(\phi)|\nabla\phi(x)|\end{aligned}$$

Using both results we can compute

$$\begin{aligned}\frac{\partial E}{\partial\phi} &= \frac{\partial\mathcal{L}}{\partial\phi} - \operatorname{div}\left(\frac{\partial\mathcal{L}}{\partial\nabla\phi}\right) \\ &= \delta(\phi) \cdot (f_1 - f_2) + \nu\delta'(\phi)|\nabla\phi(x)| - \nu \operatorname{div}\left(\delta(\phi) \cdot \frac{\nabla\phi}{|\nabla\phi|}\right)\end{aligned}$$

Further simplifying gives:

$$\begin{aligned}\operatorname{div}\left(\delta(\phi) \cdot \frac{\nabla\phi}{|\nabla\phi|}\right) =& \frac{\mathrm{d}}{\mathrm{d}x}\left(\delta(\phi) \cdot \frac{\nabla\phi}{|\nabla(\phi)|}\right) + \frac{\mathrm{d}}{\mathrm{d}y}\left(\delta(\phi) \cdot \frac{\nabla\phi}{|\nabla(\phi)|}\right) \\ =& \left(\frac{\mathrm{d}}{\mathrm{d}x}\delta(\phi)\right) \cdot \frac{\nabla\phi}{|\nabla(\phi)|} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\nabla\phi}{|\nabla(\phi)|}\right) \cdot \delta(\phi) \\ &+ \left(\frac{\mathrm{d}}{\mathrm{d}y}\delta(\phi)\right) \cdot \frac{\nabla\phi}{|\nabla(\phi)|} + \left(\frac{\mathrm{d}}{\mathrm{d}y}\frac{\nabla\phi}{|\nabla(\phi)|}\right) \cdot \delta(\phi) \\ =& \delta'(\phi) \cdot \phi_x \cdot \frac{\nabla\phi}{|\nabla(\phi)|} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\nabla\phi}{|\nabla(\phi)|}\right) \cdot \delta(\phi) \\ &+ \delta'(x) \cdot \phi_y \cdot \frac{\nabla\phi}{|\nabla(\phi)|} + \left(\frac{\mathrm{d}}{\mathrm{d}y}\frac{\nabla\phi}{|\nabla(\phi)|}\right) \cdot \delta(\phi) \\ =& \delta(\phi) \operatorname{div}\left(\frac{\nabla\phi}{|\nabla(\phi)|}\right) + \delta'(\phi) \cdot \nabla\phi \cdot \frac{\nabla\phi}{|\nabla(\phi)|} \\ =& \delta'(\phi)|\nabla\phi| + \delta(\phi) \cdot \operatorname{div}\left(\frac{\nabla\phi}{|\nabla(\phi)|}\right)\end{aligned}$$

Finally:

$$
\begin{aligned}
\frac{\partial E}{\partial \phi} &= \delta(\phi) \cdot (f_1 - f_2) + \nu\delta'(\phi)|\nabla\phi(x)| - \nu\delta'(\phi)|\nabla\phi| - \nu\delta(\phi) \cdot \operatorname{div}\left(\frac{\nabla\phi}{|\nabla(\phi)|}\right) \\
&= \delta\phi\left(f_1 - f_2 - \nu \operatorname{div}\left(\frac{\nabla\phi}{|\nabla(\phi)|}\right)\right)
\end{aligned}
$$

2.

$$
\begin{aligned}
E(\phi) &= \int g(x)|\nabla H(\phi(x))|\mathrm{d}x \\
&= \int g(x)\delta(\phi)|\nabla\phi|\mathrm{d}x
\end{aligned}
$$

With $\frac{\partial E}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \operatorname{div}\left(\frac{\partial \mathcal{L}}{\partial \nabla\phi}\right)$ this leads to:

$$
\frac{\partial \mathcal{L}}{\partial \phi} = \delta'(\phi)g(x)|\nabla\phi|
$$

and

$$
\frac{\partial \mathcal{L}}{\partial \nabla\phi} = g(x)\delta(\phi)\frac{\nabla\phi}{|\nabla(\phi)|}
$$

Combining both results and further evaluating $\operatorname{div}\left(g(x)\delta(\phi)\frac{\nabla\phi}{|\nabla(\phi)|}\right)$ gives:

$$
\begin{aligned}
\frac{\partial E}{\partial \phi} &= \frac{\partial L}{\partial \phi} - div\left(\frac{\partial L}{\partial \nabla\phi}\right) \\
&= \delta'(\phi)g(x)|\nabla\phi| - div\left(g(x)\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}\right) \\
&= \delta'(\phi)g(x)|\nabla\phi| - \delta'(\phi)g(x)\frac{\nabla\phi \cdot \nabla\phi}{|\nabla\phi|} - \delta(\phi)div\left(g(x)\frac{\nabla\phi}{|\nabla\phi|}\right) \\
&= -\delta(\phi)div\left(g(x)\frac{\nabla\phi}{|\nabla\phi|}\right)
\end{aligned}
$$