

Chapter 2

Diffusion Filtering

Variational Methods for Computer Vision

Winter 2014/15



Nonlinear Filtering

Partial Differential
Equations

Diffusion

Nonlinear &
Anisotropic Diffusion

Numerical
Implementation

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Nonlinear Filtering

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Nonlinear Filtering

- The **convolution** of an input image $f(x)$ with a kernel $G(x)$:

$$g(x) = (G * f)(x) = \int G(x')f(x - x')dx'$$

is a classical example of a **linear filter**.

- Convolutions can be **efficiently implemented** in frequency space because in frequency space the convolution corresponds to a simple (frequency-wise) product and because the Fast Fourier transform allows a quick conversion to and from frequency space.
- In practice, however, linear filters are often suboptimal. In smoothing/denoising, for example, the Gaussian smoothing **removes both noise and signal** – semantically relevant structures tend to disappear along with the noise. Instead, one would like to remove noise in an **adaptive** manner such that semantically important structures remain unaffected. In principle this could be done with a Gaussian smoothing where the filter width σ is adapted to the local structure (larger in noise areas, smaller at important edges).





- Formally this would amount to the following:

$$g(x) = \int G_{\sigma(f,x)}(x')f(x - x')dx',$$

where now the width σ of the convolution kernel G depends on the brightness values in a local neighborhood.

- It turns out that there exist other more elegant solutions to model such adaptive denoising processes by means of **Diffusion filtering**.
- The key observation is that image smoothing can be modeled with a diffusion process. In this process, the local brightness diffuses to neighboring pixels due to differences in the local concentration of grayvalue.
- Mathematically diffusion processes are represented by **partial differential equations** (PDEs).

Review: Partial Differential Equations

- A **partial differential equation (PDE)** is an equation containing the partial derivatives of a function of several variables.

Example — the wave equation:

$$\frac{\partial^2 \psi(x, t)}{\partial t^2} = c^2 \Delta \psi(x, t)$$

- For functions of a single variable we have the special case of **ordinary differential equations (ODEs)** (gewöhnliche Differentialgleichungen).

Example — the pendulum:

$$m \frac{d^2 x(t)}{dt^2} + \gamma \frac{dx(t)}{dt} + kx(t) = 0$$

- Many natural phenomena can be modeled by partial differential equations. In most cases, one can derive the respective equation from a few basic principles. A **solution** of a differential equation is a function for which the differential equation is true.



Analytical Solutions

- A few PDEs can be solved **analytically**, i.e. the solution can be written in closed form.
- Example — The wave equation (in 1D):

$$\frac{\partial^2 \psi(x, t)}{\partial t^2} = c^2 \frac{\partial^2 \psi(x, t)}{\partial x^2}$$

has the (not necessarily unique) solution:

$$\psi(x, t) = \sin(x - ct)$$

- If solutions are not unique one can impose additional assumptions **boundary conditions** or **initial conditions**, for example $\psi(x, 0) = \psi_0(x)$.
- Example — The harmonic oscillator (without friction):

$$m \frac{d^2 x(t)}{dt^2} + kx(t) = 0$$

has the (generally not unique) solution:

$$x(t) = \sin(\omega t), \quad \text{with } \omega = \sqrt{k/m}.$$



The Diffusion Equation

- Diffusion is a physical process which aims at minimizing differences in the spatial concentration $u(x, t)$ of a substance.
- This process can be described by two basic equations:
 - Fick's law states that concentration differences induce a flow j of the substance in direction of the negative concentration gradient:

$$j = -g\nabla u$$

The **diffusivity** g describes the speed of the diffusion process.

- The continuity equation

$$\partial_t u = -\operatorname{div} j$$

where $\operatorname{div} j \equiv \nabla j \equiv \partial_x j_1 + \partial_y j_2$ is called the **divergence** of the vector j .

- Inserting one into the other leads to the diffusion equation:

$$\partial_t u = \operatorname{div} (g \cdot \nabla u)$$



Solution of the Linear Diffusion Equation



The one-dimensional linear diffusion equation ($g = 1$)

$$\partial_t u = \partial_x^2 u.$$

with initial condition

$$u(x, t = 0) = f(x)$$

has the unique solution:

$$u(x, t) = (G_{\sqrt{2t}} * f)(x) = \int_{-\infty}^{\infty} G_{\sqrt{2t}}(x - x') f(x') dx',$$

where

$$G_{\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}},$$

is a Gaussian kernel of width $\sigma = \sqrt{2t}$.



- The above result implies that smoothing of an image $f : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ with Gaussian kernels of increasing width σ can be realized through a diffusion process of the form

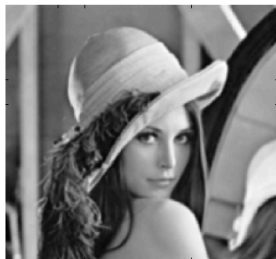
$$\begin{cases} \partial_t u(x, t) = \Delta u \\ u(x, 0) = f(x) \quad \forall x \in \Omega \\ \partial_n u|_{\partial\Omega} = \langle \nabla u, n \rangle|_{\partial\Omega} = 0 \end{cases}$$

- The latter boundary condition states that the derivative of the brightness function u along the normal n at the image boundary $\partial\Omega$ must vanish. This assures that **no brightness will leave or enter the image**, i.e. the average brightness will be preserved.
- With increasing time t the solution $u(x, t)$ of this process will correspond to increasingly smoothed versions of the original image $f(x)$.

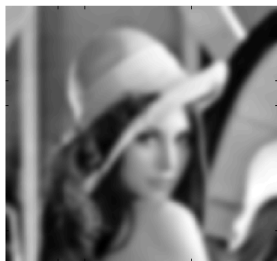
Image Smoothing via Diffusion



Lena original



diffusion $t = 2$



diffusion $t = 20$



diffusion $t = 100$





- General diffusion equation:

$$\partial_t u = \operatorname{div}(g \nabla u)$$

- For $g = 1$ (or $g = \text{const.} \in \mathbb{R}$) the diffusion process is called **linear, isotropic** and **homogeneous**.
- If the diffusivity g is space-dependent, i.e. $g = g(x)$, the process is called an **inhomogeneous diffusion**.
- If the diffusivity depends on u , i.e. $g = g(u)$, then it is called a **nonlinear diffusion** because then the equation is no longer linear in u .
- If the diffusivity g is matrix-valued then the process is called an **anisotropic diffusion**. A matrix-valued diffusivity leads to processes where the diffusion is different in different directions.
- Note: In the literature this terminology is not used consistently.



- Idea: Less diffusion (smoothing) in locations of strong edge information.
- Gradient norm $|\nabla u| = \sqrt{u_x^2 + u_y^2}$ serves as **edge indicator**
- Diffusivity should decrease with increasing $|\nabla u|$. For example (**Perona & Malik, *Scale Space and Edge Detection using Anisotropic Diffusion, PAMI 1990***):

$$g(|\nabla u|) = \frac{1}{\sqrt{1 + |\nabla u|^2 / \lambda^2}}$$

- $\lambda > 0$ is called a **contrast parameter**. Areas where $|\nabla u| \gg \lambda$ will not be affected much by the diffusion process.
- The Perona-Malik model had a huge impact in image processing because it allowed a better edge detection than classical edge detectors (such as the Canny edge detector).

Implementation with Finite Differences

- Nonlinear diffusion equation:

$$\partial_t u = \partial_x (g(|\nabla u|)\partial_x u) + \partial_y (g(|\nabla u|)\partial_y u)$$

- Discretize the operators as:

$$\partial_t u \approx \frac{u_{ij}^{t+1} - u_{ij}^t}{\tau}$$

and

$$\begin{aligned}\partial_x (g\partial_x u) &\approx \left((g\partial_x u)_{i+1/2,j}^t - (g\partial_x u)_{i-1/2,j}^t \right) \\ &\approx \left(g_{i+1/2,j}^t (u_{i+1,j}^t - u_{ij}^t) - g_{i-1/2,j}^t (u_{ij}^t - u_{i-1,j}^t) \right)\end{aligned}$$

where $g_{i+1/2,j} = \sqrt{g_{i+1,j}g_{ij}}$ assures that no diffusion takes place as soon as g is zero at one of the two pixels.

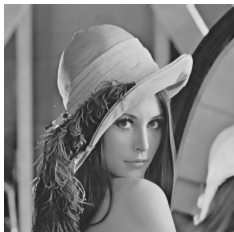
- Insert, solve for u_{ij}^{t+1} and iterate in t .
- Source: **J. Weickert**, *Anisotropic Diffusion in Image Processing*.



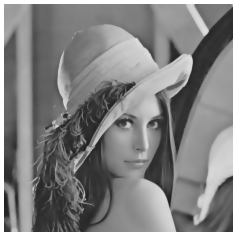
Nonlinear Diffusion



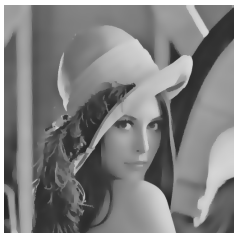
Lena original



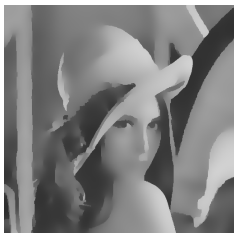
diffusion $t = 9$



diffusion $t = 25$



diffusion $t = 100$



diffusion $t = 400$



diffusion $t = 900$

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