Chapter 9
Convex Relaxation Methods

Variational Methods for Computer Vision

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Overview

Convexity and Globally Optimal Solutions

In the last section, we saw that level set methods allow to minimize geometric optimization problems in such a way that the represented curve or surface can undergo topological changes like splitting or merging. Due to the implicit representation of the geometry, the resulting optimization process can take into account a larger space of feasible solutions (including shapes of different topology).

Nevertheless, respective energies are not convex and thus level set methods typically only determine local optima. While the computed solutions are often good, we generally do not have a performance guarantee, i.e. we do not know how far we are from the optimal solution.

Starting in 2005, researchers have proposed novel variational approaches which are aimed at approximating the original energies with convex functionals. Rather than minimizing the original energy locally, they minimize an approximation of the original energy globally. How far this framework can be extended to the kinds of energies arising in computer vision is among the major challenges in current research.

Convex Two-Region Segmentation

Let us start with a Mumford-Shah-like model with two regions (foreground / background) and fixed color models:

$$\min_{\Omega_1} \int_{\Omega_1} f_1(x)\,dx + \int_{\Omega-\Omega_1} f_2(x)\,dx + \nu\,|\partial\Omega_1|,$$

with integrals over $\Omega_1 \subset \Omega$ and its complement $\Omega - \Omega_1$.

The integrands may for example arise

- from a Gaussian color model for each region:

$$f_i(x) = \frac{(I(x)-\mu_i)^2}{2\sigma_i^2} + \log \sigma_i$$

- or from a general color distribution p_i:

$$f_i(x) = -\log p_i(I(x))$$

The term $|\partial\Omega_1|$ denotes the length of the boundary $\partial\Omega_1$.

Convex Two-Region Segmentation

In Chan, Esedoglu, Nikolova, *Trans. on Image Proc. 2006*, the authors propose to encode the two-region segmentation by a binary indicator function

$$u : \Omega \to \{0,1\}, \quad u(x) = \begin{cases} 1, & \text{if } x \in \Omega_1 \\ 0, & \text{else} \end{cases}$$

In terms of u, the segmentation problem is

$$E(u) = \int_\Omega f_1(x)\, u(x)\, dx + \int_\Omega f_2(x)\, \big(1 - u(x)\big)\, dx + \nu \int_\Omega |\nabla u(x)|\, dx.$$

It is related to the Chan-Vese model by associating $u \equiv H(\phi)$.

The above functional is convex in u because the first two terms are linear in u and the total variation of u is also convex.

The overall optimization problem is not convex because the space of binary functions u is not a convex space: Convex combinations of binary functions are typically no longer binary.

Convex Two-Region Segmentation

The two-region segmentation problem is defined over the space $\mathrm{BV}(\Omega; \{0, 1\})$, the space of functions of bounded variation, i.e. functions u for which the total variation $\mathrm{TV}(u)$ is finite.

Relaxation denotes the technique of simply dropping certain constraints from the overall optimization problem. Convex relaxation means that upon relaxation the problem becomes convex.

Chan et al. (2006) convexify the two-region segmentation problem by simply dropping the constraint that u must be binary. They allow u to take on values in the entire interval $[0, 1]$, which is the convex hull of the original domain:

$$\min_{u \in \mathrm{BV}(\Omega;[0,1])} E(u).$$

By construction, this is a convex optimization problem. The hard labeling of each pixel as 0 or 1 is replaced by a soft labeling of each pixel with some value between 0 and 1.

Convex Relaxation & Global Optimality

In general, the optimum of the relaxed problem

$$u^\star = \arg \min_{u \in \mathrm{BV}(\Omega;[0,1])} E(u)$$

is not binary. A binary function is obtained by thresholding:

$$1_{u^\star > \theta}(x) = \begin{cases} 1, & \text{if } u^\star(x) > \theta \\ 0, & \text{else} \end{cases}$$

Such relaxation techniques can be applied to many optimization problems. In general, one loses optimality as the thresholded solution is typically not an optimum for the original binary labeling problem.

Surprisingly, this is not the case for the functional considered here. More specifically, one can show that the thresholded solution $1_{u^\star > \theta}$ has the same energy as the relaxed solution $u^\star$. As a consequence, it is indeed a global optimum of the original binary labeling problem.

Thresholding Theorem

Let

$$u^\star = \arg \min_{u \in \mathrm{BV}(\Omega;[0,1])} \int f u + |\nabla u| \, dx,$$

be a global minimizer of the relaxed problem with an arbitrary function f. Then the function $1_{u^\star > \theta}$ is a global minimizer of the corresponding binary optimization problem for any threshold value $\theta \in (0,1)$.

The proof of this theorem makes use of the layer cake formula

$$u(x) = \int_0^1 1_{u>\theta}(x) d\theta$$

and the coarea formula

$$\mathrm{TV}(u) = \int_\Omega |\nabla u| \, dx = \int_0^1 \int_\Omega |\nabla 1_{u>\theta}(x)| \, dx \, d\theta,$$

stating that $\mathrm{TV}(u)$ equals the sum of the lengths of all level lines of u.

Proof by Contradiction

Using the layer cake and coarea formula, we can write

$$E(u) = \int_\Omega fu + |\nabla u| \, dx = \int_0^1 \int_\Omega f 1_{u>\theta} + |\nabla 1_{u>\theta}| \, dx \, d\theta = \int_0^1 E(1_{u>\theta}) \, d\theta.$$

Assume that the thresholded version $1_{u^\star > \theta_0}$ is not the optimum of the binary problem for some value $\theta_0 \in (0, 1)$, i.e. there exists a set $\Sigma \subset \Omega$ with $E(1_\Sigma) < E(1_{u^\star > \theta_0})$.

Then due to continuity of the energy, there exists some $\epsilon > 0$ with:

$$E(1_\Sigma) < E(1_{u^\star > \theta}) \quad \forall\, \theta \in (\theta_0 - \epsilon, \theta_0 + \epsilon).$$

As a consequence, we get:

$$E(1_\Sigma) = E(1_\Sigma) \int_0^1 d\theta = \int_0^1 E(1_\Sigma) \, d\theta < \int_0^1 E(1_{u^\star > \theta}) \, d\theta = E(u^\star).$$

This contradicts the assumption that $u^\star$ is a global minimizer.

Some Comments

The thresholding theorem is somewhat surprising at first glance. It does not hold for general binary labeling problems. Otherwise, one could solve all sorts of NP hard binary optimization problems by simply embedding them in a continuous space and solving them there.

The main ingredient in the proof is that the energy of any function u can be obtained by simply summing the energies of all its upper level sets $1_{u>\theta}$. If u is optimal then so are all its upper level sets. They must in fact have the same energy.

A closer look reveals that this property of the energy being decomposable into energies of the upper level sets is tightly related to respective properties of the total variation and its geometric interpretation being the sum of the lengths of all level lines. It is one of the many properties which make the total variation extremely popular in the field of optimization.

In the spatially discrete setting, corresponding binary labeling problems can be solved in polynomial time using the min-cut / max-flow duality of Ford and Fulkerson (1962).

General Definition of Total Variation

So far, we worked with a definition of total variation which is not differentiable and which only applies to differentiable functions.

A remedy is given by introducing a dual variable $\xi \in \mathbb{R}^2$ ("Xi"):

$$|\nabla u| = \sup_{|\xi| \leq 1} \xi \cdot \nabla u,$$

where the supremum is attained at $\xi = \frac{\nabla u}{|\nabla u|}$ if $\nabla u \neq 0$.

It allows to generalize the total variation to a differentiable expression which also applies to discontinuous functions u:

$$\mathrm{TV}(u) := \sup_{\xi \in \mathcal{K}} \int u \,\mathrm{div}\xi \, dx \overset{u \text{ diffable}}{=} \sup_{\xi \in \mathcal{K}} \int \xi \nabla u \, dx = \int |\nabla u| \, dx,$$

with the dual variable ξ being a differentiable vector field with compact support (i.e. $\xi = 0$ at the boundary), constrained to the unit disc at every point $x \in \Omega$:

$$\mathcal{K} = \left\{ \xi \in C_c^1(\Omega; \mathbb{R}^2) \,\middle|\, |\xi(x)| \leq 1 \;\forall x \in \Omega \right\}.$$

Minimization with Primal Dual Algorithm

The two-region segmentation with known color models can be solved by thresholding the solution of the relaxed (convex) problem which is of the form

$$\min_{u \in C} \int f u \, dx + \mathrm{TV}(u) = \min_{u \in C} \sup_{\xi \in \mathcal{K}} \int f u + u \, \mathrm{div} \xi \, dx,$$

where $C = \mathrm{BV}(\Omega; [0, 1])$.

An efficient algorithm for minimizing this saddle point problem was proposed in Pock, Cremers, Chambolle, Bischof, ICCV 2009. It amounts to an alternating projected gradient descent / ascent with an extrapolation step:

$$\begin{cases} \xi^{n+1} = \Pi_{\mathcal{K}} \left(\xi^n - \sigma \nabla \bar{u}^n \right), \\ u^{n+1} = \Pi_C \left(u^n - \tau (\mathrm{div} \, \xi^{n+1} + f) \right), \\ \bar{u}^{n+1} = u^{n+1} + (u^{n+1} - u^n) = 2u^{n+1} - u^n, \end{cases}$$

where $\Pi_{\mathcal{K}}$ and Π_C denote the back-projections onto $\mathcal{K}$ and C. It provably converges for sufficiently small step sizes σ and τ.

Back Projection onto Convex Sets

For the primal variable u the projection onto the set $C = \text{BV}(\Omega; [0,1])$ is done by clipping:

$$(\Pi_C u)(x) = \min\left\{1, \max\{0, u(x)\}\right\} = \begin{cases} u(x), & \text{if } u(x) \in [0,1] \\ 1, & \text{if } u(x) > 1 \\ 0, & \text{if } u(x) < 0 \end{cases}$$

For the dual variable ξ projection onto the unit disk $\mathcal{K}$ is done as follows:

$$(\Pi_{\mathcal{K}} \xi)(x) = \frac{\xi(x)}{\max\{1, |\xi(x)|\}}.$$

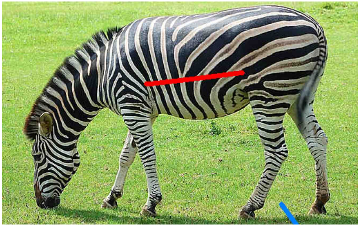
Both of these projections can obviously be done in closed form.

For more complicated convex constraint sets, the constraints can typically no longer be imposed in closed form. In such cases one reverts to alternating several simple projections or to Lagrange multipliers and related approaches.

Interactive Two-Region Segmentation

Algorithm:

- Determine color distributions $p_{obj}(I)$ and $p_{bg}(I)$ for object and background from user scribbles.
- Compute for all pixels $f(x) = \log \frac{p_{obj}(I(x))}{p_{bg}(I(x))}$.
- Solve the relaxed convex problem and threshold the solution.

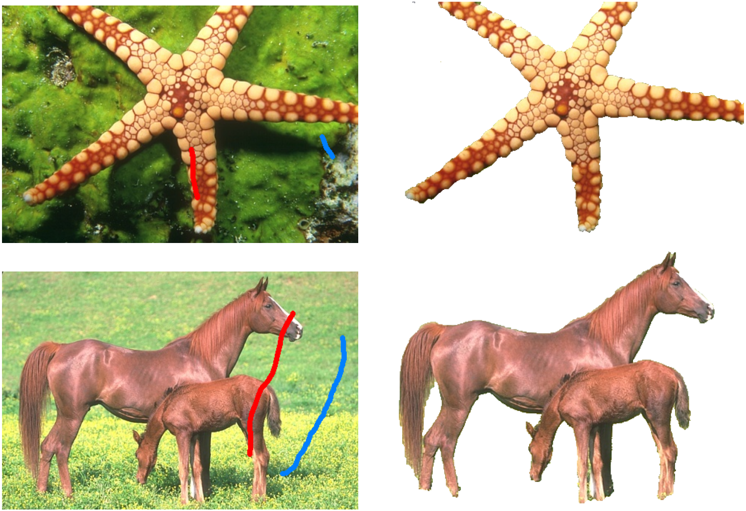


Input

Segmentation

Unger, Pock, Cremers, Bischof, TVSeg, *BMVC 2008*

Interactive Two-Region Segmentation



Input Segmentation

Unger, Pock, Cremers, Bischof, TVSeg, *BMVC 2008*

Convex Multi-Region Segmentation

The multiregion segmentation problem is of the form

$$\min_{\Omega_1,\dots,\Omega_n} \sum_{i=1}^{n} \int_{\Omega_i} f_i(x)\,dx + \frac{1}{2}|\partial\Omega_i|,$$

with arbitrary data terms f_i, for example $f_i = -\log p_i(I(x))$.

In Chambolle, Cremers, Pock, 2008, a convex formulation was proposed using the indicator function for region Ω_i:

$$v_i(x) = 1_{\Omega_i}(x) = \begin{cases} 1, & \text{if } x \in \Omega_i \\ 0, & \text{else} \end{cases}$$

Then the multi-region segmentation problem is equivalent to

$$\min_{v\in\mathcal{B}} \sum_{i=1}^{n} \int_{\Omega} f_i(x)\,v_i + \frac{1}{2}|\nabla v_i|\,dx$$

with $\mathcal{B} = \Big\{(v_1,\dots,v_n) \in \mathrm{BV}(\Omega;\{0,1\})^n \;\Big|\; \sum_i v_i(x) = 1 \;\forall x \in \Omega\Big\}$.

Convex Multi-Region Segmentation

In Chambolle, Cremers, Pock, 2008, it is shown that

$$\min_{v \in \mathcal{B}} \sum_{i=1}^{n} \int_{\Omega} f_i(x)\, v_i + \frac{1}{2}|\nabla v_i|\, dx = \min_{v \in \mathcal{B}} \sup_{p \in \mathcal{K}} \sum_{i=1}^{n} \int_{\Omega} f_i(x)\, v_i + v_i \operatorname{div} p_i\, dx$$

with the convex set

$$\mathcal{K} = \left\{ (p_1, \dots, p_n)^\top \in \mathbb{R}^{n \times 2} \;\middle|\; |p_i - p_j| \leq 1 \;\forall i, j \right\}.$$

Intuitively, the dual variables p_i account for the discontinuities in the labeling v_i. The coupling constraint $|p_i - p_j| \leq 1$ implies that the transition from label i to label j should not count more than 1 (in fact, exactly 1 in the supremum).

As in the two-region case of Chan et al., 2006, one obtains a convex problem by dropping the binarity constraint:

$$v \in \mathcal{B}_{rel} = \left\{ (v_1, \dots, v_n) \in \mathrm{BV}(\Omega; [0,1])^n \;\middle|\; \sum_i v_i(x) = 1 \;\forall x \in \Omega \right\}.$$

Convex Multi-Region Segmentation

Algorithm:

- Specify Gaussian color models $p_i(I)$ for each region i.
- Compute for all pixels $f_i(x) = -\log p_i(I(x))$.
- Solve the relaxed convex problem and binarize the solution.

Note:

There is no thresholding theorem for the multi-region case. While the relaxed problem can be solved optimally, the subsequent binarization does not assure an optimal solution to the original labeling problem.

It provides approximate solutions to the original problem which are independent of initialization.

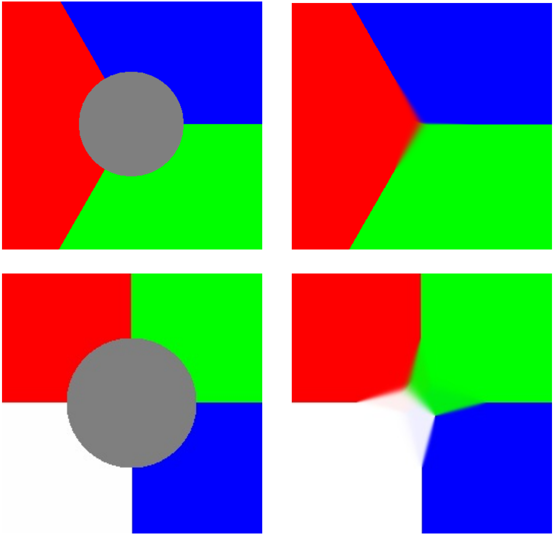
Since the multilabel problem in its spatially discrete form is NP hard, it is unlikely that such a thresholding theorem exists.

Convex Multi-Region Segmentation

Input Segmentation

Chambolle, Cremers, Pock, 2008, SIAM Imaging Sci. 2012

Minimal Boundary Inpainting



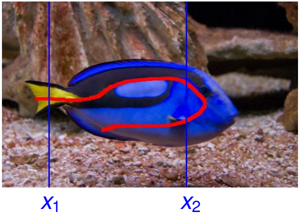
missing colors inpainted

Chambolle, Cremers, Pock, 2008, SIAM Imaging Sci. 2012

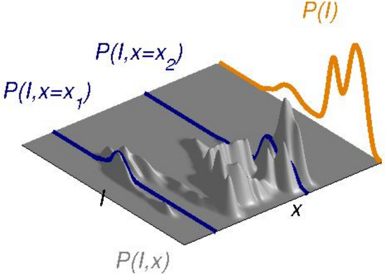
Interactive Multi-Region Segmentation

Algorithm:

- Determine a space-varying color distribution $p_i(I \mid x)$ for each region i from user scribbles.
- Compute for all pixels $f_i(x) = -\log p_i(I(x)|x)$.
- Solve the relaxed convex problem and threshold the solution.



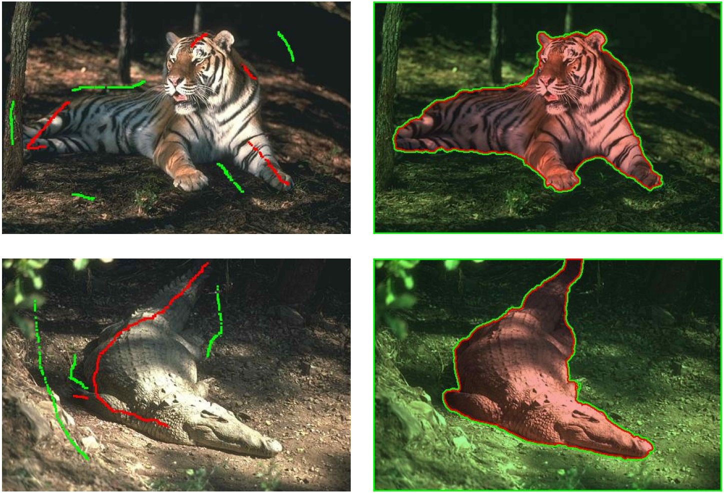
Input & scribbles



Space-varying color likelihoods

Nieuwenhuis, Cremers, PAMI 2013

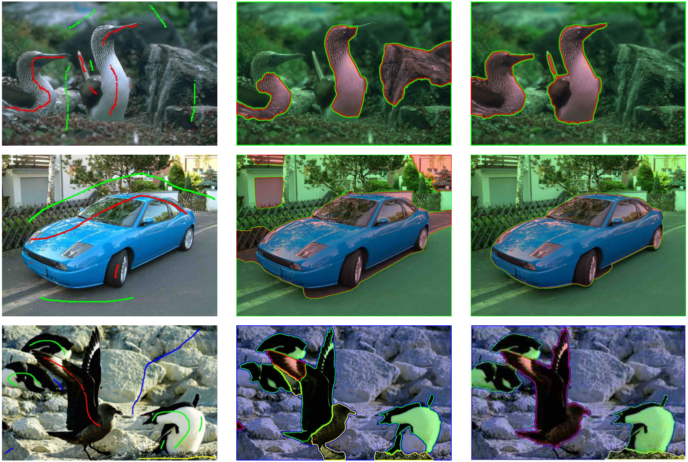
Interactive Multi-Region Segmentation



Input & scribbles Segmentation

Nieuwenhuis, Cremers, PAMI 2013

Interactive Multi-Region Segmentation



Input & scribbles | Segm. with $p(I)$ | Segm. with $p(I|x)$

Nieuwenhuis, Cremers, PAMI 2013

Code available at

`http://vision.in.tum.de/_media/data/software/cudamultilabeloptimization-v1.0.zip`.