

# Variational Methods for Computer Vision: Exercise Sheet 2

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Exercise: October 30, 2014

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## Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. A multidimensional filter is called separable, if it can be decomposed in one dimensional filter operations. Prove that the convolution of an image  $f$  with a Gaussian kernel  $K$  of standard deviation  $\sigma > 0$ :

$$K(x, y) := \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right).$$

can be written as the convolution with two one-dimensional filters:

$$k_1(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right). \quad \text{and} \quad k_2(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right).$$

Hence :

$$f(x, y) * K(x, y) = (f(x, y) * k_1(x)) * k_2(y)$$

Explain why the separability of a filter is a desirable property.

2. Let  $f \in C^2(\Omega; \mathbb{R})$  with  $\Omega \subset \mathbb{R}^2$  be a real valued function and let  $R \in SO(2)$  be a rotation matrix. Let  $\tilde{f}(x) := f(R \cdot x)$  be a rotated version of  $f$ . Prove that the magnitude of the gradient and the Laplace operator are rotationally covariant by showing the following identities:

- (a)  $R\nabla\tilde{f}(x) = (\nabla f)(R \cdot x)$
- (b)  $\|\nabla\tilde{f}(x)\| = \|(\nabla f)(R \cdot x)\|$
- (c)  $\Delta\tilde{f}(x) = (\Delta f)(R \cdot x)$

*Reminder:*  $R \in SO(2)$  denotes  $2 \times 2$  matrices with  $\det(R) = 1$  and  $R^T R = \mathbb{1}$ . and can be written as

$$R = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$

such that  $\alpha \in [0, 2\pi]$ .

3. The general diffusion equation for a function can be written as follows:

$$\partial_t u(x, t) = \operatorname{div}(g(u, x)\nabla u(x, t))$$

Where  $u \in C^2(\Omega \times \mathbb{R}_0^+; \mathbb{R})$  with  $\Omega \subset \mathbb{R}^2$  describes the complete diffusion process and solves the partial differential equation. Prove the following identities:

- (a) *linear homogeneous diffusion:*

$$\operatorname{div}(g\nabla u(x)) = g\Delta u(x) \quad \text{s.t.} \quad g \in \mathbb{R}$$

- (b) *linear inhomogeneous diffusion:*

$$\operatorname{div}(g(x)\nabla u(x)) = g(x)\Delta u(x) + \langle \nabla g(x), \nabla u(x) \rangle \quad \text{s.t.} \quad g \in C^1(\Omega; \mathbb{R})$$

## Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. In the last exercise we implemented an image convolution with the Gaussian kernel using the straightforward definitions. Considering the results from the first theoretical exercise, write a Matlab script that implements a separable convolution with a Gaussian kernel.
2. Download the archive file `vmcv_ex01.zip` from the homepage and unzip it in your home folder. Use the template file `difusion_filter.m` for a non linear diffusion filter and complete the missing code at line 58. Test the script on the image `lena.png`.
3. Create a video using the `avifile` command and compare the results.

### Matlab-Tutorials:

<http://www.math.utah.edu/lab/ms/matlab/matlab.html>

<http://www.math.ufl.edu/help/matlab-tutorial/>

<http://www.glue.umd.edu/~nsw/ench250/matlab.htm>