Variational Methods for Computer Vision: Solution Sheet 2

Exercise: October 30, 2014

Part I: Theory

1.

$$\begin{aligned}(f(x,y)*k_1(x))*k_2(y) &= \int\left(\int k_1(k)f(x-k,y-l)dk\right)k_2(l)dl\\ &= \int\int f(x-k,y-l)k_1(k)k_2(l)dkdl\\ &= \int\int f(x-k,y-l)\frac{1}{2\pi\sigma^2}e^{-\frac{k^2+l^2}{2\sigma^2}}dkdl\\ &= \int\int f(x-k,y-l)K(k,l)dkdl\\ &= f(x,y)*K(x,y)\end{aligned}$$

2. (a)

$$\begin{aligned}\nabla\widetilde{f}(x) &= \nabla(f\circ R)(x)\\ &= (D_{Rx}f\circ D_xR)^T\\ &= (D_{Rx}f\circ R)^T\\ &= R^TD_{Rx}f^T\\ &= R^T\nabla f(Rx)\end{aligned} \tag{1}$$

Thus:

$$\begin{aligned}R\nabla\widetilde{f}(x) &= RR^T\nabla f(Rx)\\ &= \nabla f(Rx)\end{aligned}$$

(b)

$$\begin{aligned}|\nabla\widetilde{f}(x)| &\overset{(1)}{=} |R^T\nabla f(Rx)|\\ &= \sqrt{(R^T\nabla f(Rx))^T(R^T\nabla f(Rx))}\\ &= \sqrt{\nabla f(Rx)^TRR^T\nabla f(Rx)}\\ &= \sqrt{\nabla f^T\nabla f}\\ &= |\nabla f(Rx)|\end{aligned}$$

(c)

$$\begin{aligned}\Delta\widetilde{f}(x) &= \text{div}(\underbrace{\nabla\widetilde{f}(x)}_{R^T\nabla f(Rx)})\\ &= \text{div}\begin{pmatrix}\cos(\alpha)\widetilde{f}_x+\sin(\alpha)\widetilde{f}_y\\ -\sin(\alpha)\widetilde{f}_x+\cos(\alpha)\widetilde{f}_y\end{pmatrix} = \widetilde{f}_{xx}+\widetilde{f}_{yy}\end{aligned}$$

Thus:

$$\begin{aligned}\widetilde{f}_{xx} &= \partial_x \left(\cos(\alpha)\widetilde{f}_x + \sin(\alpha)\widetilde{f}_y\right) \\ &= \partial_x \cos(\alpha)\widetilde{f}_x + \partial_x \sin(\alpha)\widetilde{f}_y \\ &= \cos^2(\alpha) f_{xx} + \cos(\alpha)\sin(\alpha) f_{xy} + \cos(\alpha)\sin(\alpha) f_{yx} + \sin^2(\alpha) f_{yy}\end{aligned}$$

And:

$$\begin{aligned}\widetilde{f}_{yy} &= \partial_y \left(-\sin(\alpha)\widetilde{f}_x + \cos(\alpha)\widetilde{f}_y)\right) \\ &= \sin^2(\alpha) f_{xx} - \cos(\alpha)\sin(\alpha) f_{xy} - \cos(\alpha)\sin(\alpha) f_{yx} + \cos^2(\alpha) f_{yy}\end{aligned}$$

Therefore:

$$\begin{aligned}\widetilde{f}_{xx} + \widetilde{f}_{yy} &= f_{yy}(\underbrace{\sin^2(\alpha) + \cos^2(\alpha)}_{=1}) + f_{xx}(\underbrace{\sin^2(\alpha) + \cos^2(\alpha)}_{=1}) \\ &= \Delta\left(f(Rx)\right)\end{aligned}$$

3. (a)

$$\begin{aligned}\operatorname{div}\left(g\nabla u(x)\right) &= \frac{\partial}{\partial x} g \frac{\partial}{\partial x} u(x) + \frac{\partial}{\partial y} g \frac{\partial}{\partial y} u(x) \\ &= g\Delta u(x)\end{aligned}$$

(b)

$$\begin{aligned}\operatorname{div}\left(g(x)\nabla u(x)\right) &= \frac{\partial}{\partial x}\left(g(x)\frac{\partial}{\partial x}u(x)\right) + \frac{\partial}{\partial y}\left(g(x)\frac{\partial}{\partial y}u(x)\right) \\ &= \frac{\partial^2}{\partial x^2}u(x)g(x) + \frac{\partial^2}{\partial y^2}u(x)g(x) + \frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}u(x) + \frac{\partial}{\partial y}g(x)\frac{\partial}{\partial y}u(x)) \\ &= g(x)\Delta u(x) + \langle \nabla g(x), \nabla u(x)\rangle\end{aligned}$$