

Variational Methods for Computer Vision: Exercise Sheet 5

Exercise: November 20, 2014

Part I: Theory

1. We proceed accordingly to the lecture:

$$\begin{aligned}\left. \frac{\partial E(u)}{\partial u} \right|_h &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (E(u + \varepsilon h) - E(u)) \\&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_a^b L(u + \varepsilon h, u' + \varepsilon h', u'' + \varepsilon h'') - L(u, u', u'') \, dx \\&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_a^b L(u, u', u'') + \frac{\partial L}{\partial u} \varepsilon h + \frac{\partial L}{\partial u'} \varepsilon h' + \frac{\partial L}{\partial u''} \varepsilon h'' + \mathcal{O}(\varepsilon^2) - L(u, u', u'') \, dx \\&= \int_a^b \frac{\partial L}{\partial u} h + \frac{\partial L}{\partial u'} h' + \frac{\partial L}{\partial u''} h'' \, dx \\&= \int_a^b \frac{\partial L}{\partial u} h \, dx - \int_a^b h \frac{d}{dx} \frac{\partial L}{\partial u'} \, dx - \int_a^b \frac{d}{dx} \frac{\partial L}{\partial u''} h' \, dx + \left[h \frac{\partial L}{\partial u'} \right]_a^b + \left[h' \frac{\partial L}{\partial u''} \right]_a^b \\&= \int_a^b h \left(\frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u'} + \frac{d^2}{dx^2} \frac{\partial L}{\partial u''} \right) \, dx - \left[h \frac{d}{dx} \frac{\partial L}{\partial u''} \right]_a^b + \left[h \frac{\partial L}{\partial u'} \right]_a^b + \left[h' \frac{\partial L}{\partial u''} \right]_a^b.\end{aligned}$$

2.

$$\begin{aligned}\left. \frac{\partial E(u)}{\partial u} \right|_h &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (E(u + \varepsilon h) - E(u)) \\&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} L(u + \varepsilon h, \nabla(u + \varepsilon h)) - L(u, \nabla u) \, dx \\&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} L(u, \nabla u) + \frac{\partial L}{\partial u} \varepsilon h + \frac{\partial L}{\partial u_x} \varepsilon \frac{\partial h}{\partial x} + \frac{\partial L}{\partial u_y} \varepsilon \frac{\partial h}{\partial y} + \frac{\partial L}{\partial u_z} \varepsilon \frac{\partial h}{\partial z} + \mathcal{O}(\varepsilon^2) - L(u, \nabla u) \, dx \\&= \int_{\Omega} \frac{\partial L}{\partial u} h \, dx + \int_{\Omega} \left\langle \nabla h, \left(\frac{\partial L}{\partial u_x} \quad \frac{\partial L}{\partial u_y} \quad \frac{\partial L}{\partial u_z} \right)^T \right\rangle \, dx \\&= \int_{\Omega} \frac{\partial L}{\partial u} h \, dx - \int_{\Omega} \left\langle h, \operatorname{div} \left(\frac{\partial L}{\partial u_x} \quad \frac{\partial L}{\partial u_y} \quad \frac{\partial L}{\partial u_z} \right)^T \right\rangle \, dx \\&\quad + \int_{\partial \Omega} h \left\langle \left(\frac{\partial L}{\partial u_x} \quad \frac{\partial L}{\partial u_y} \quad \frac{\partial L}{\partial u_z} \right)^T, n \right\rangle \, d\partial \Omega \\&= \int_{\Omega} h \left(\frac{\partial L}{\partial u} - \left(\frac{\partial}{\partial x} \frac{\partial L}{\partial u_x} + \frac{\partial}{\partial y} \frac{\partial L}{\partial u_y} + \frac{\partial}{\partial z} \frac{\partial L}{\partial u_z} \right) \right) \, dx \\&\quad + \int_{\partial \Omega} h \left(\frac{\partial L}{\partial u_x} n_x + \frac{\partial L}{\partial u_y} n_y + \frac{\partial L}{\partial u_z} n_z \right) \, d\partial \Omega.\end{aligned}$$

Hence we can write the Euler-Lagrange equation as

$$\begin{aligned}\frac{\partial L}{\partial u} - \operatorname{div} \left(\frac{\partial L}{\partial \nabla u} \right) &= 0, & \text{on } \Omega, \\ \left\langle \frac{\partial L}{\partial \nabla u}, \nu \right\rangle &= 0, & \text{on } \partial\Omega,\end{aligned}$$

where ν denotes the normal vector on the boundary $\partial\Omega$.

3. (a) We have that $L(u, \nabla u) = \sqrt{u_x^2 + u_y^2}$ and

$$\begin{aligned}\frac{\partial L}{\partial u_x} &= \frac{u_x}{\sqrt{u_x^2 + u_y^2}}, \\ \frac{\partial L}{\partial u_y} &= \frac{u_y}{\sqrt{u_x^2 + u_y^2}}.\end{aligned}$$

Thus the Euler-Lagrange equation is given as the following:

$$\frac{\partial E}{\partial u} = \frac{\partial L}{\partial u} - \operatorname{div} \left(\frac{\partial L}{\partial \nabla u} \right) = -\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right).$$

- (b) Similarly to the previous exercise we arrive for $L(u, \nabla u) = \sqrt{(\nabla u)^\top D \nabla u}$ at the following Euler-Lagrange equation:

$$\frac{\partial E}{\partial u} = \frac{\partial L}{\partial u} - \operatorname{div} \left(\frac{\partial L}{\partial \nabla u} \right) = -\operatorname{div} \left(\frac{(D + D^\top) \nabla u}{2\sqrt{(\nabla u)^\top D \nabla u}} \right).$$