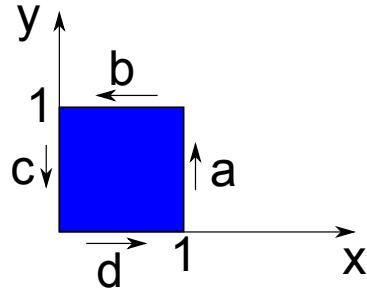


# Variational Methods for Computer Vision: Solution Sheet 7

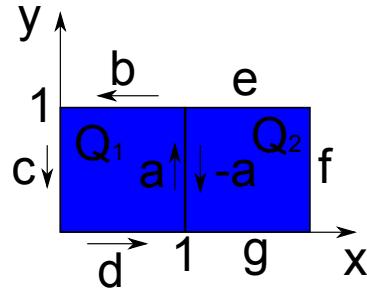
Exercise: January 11, 2014

## Part I: Theory



1. (a)

$$\begin{aligned} \int_Q v_x(x, y) - u_y(x, y) dx dy &= \int_0^1 \int_0^1 v_x(x, y) - u_y(x, y) dx dy \\ &= \int_0^1 \int_0^1 v_x(x, y) dx dy - \int_0^1 \int_0^1 u_y(x, y) dy dx \\ &= \int_0^1 v(x, y)|_{x=0}^{x=1} dy - \int_0^1 v(x, y)|_{y=0}^{y=1} dx \\ &= \int_0^1 v(1, y) - v(0, y) dy - \int_0^1 v(x, 1) - v(x, 0) dx \\ &= \int_0^1 v(1, y) dy - \int_0^1 v(0, y) dy - \int_0^1 v(x, 1) dx + \int_0^1 v(x, 0) dx \\ &= \underbrace{\int_0^1 v(1, y) dy}_a + \underbrace{\int_1^0 v(0, y) dy}_c + \underbrace{\int_0^1 v(x, 1) dx}_b + \underbrace{\int_0^1 v(x, 0) dx}_d \\ &= \int_{\partial Q} v ds \end{aligned}$$



(b)

$$\begin{aligned}
 & \int_{Q_1} v_x(x, y) - u_y(x, y) dx dy + \int_{Q_2} v_x(x, y) - u_y(x, y) dx dy \\
 &= \int_a^b v(x, y) dy \int_c^d v(x, y) dy + \int_b^d v(x, y) dx + \int_d^g v(x, y) dx \\
 &\quad - \int_a^b v(x, y) dy + \int_e^f v(x, y) dx + \int_f^g v(x, y) dx
 \end{aligned}$$

2. (a) The curvature  $\kappa$  of a circle with radius  $r$  is  $\kappa = \frac{1}{r}$ . We can use this fact in calculating the Euler-Lagrange equations for the 2 different cases.

$r > 1$ :

$$\begin{aligned}
 u_{\text{outer}} &= 0 \\
 u_{\text{inner}} &= \frac{\pi}{\pi r^2} = \frac{1}{r^2}
 \end{aligned}$$

This leads to following Euler-Lagrange equation:

$$\begin{aligned}
 & (I - u_{\text{outer}})^2 - (I - u_{\text{inner}})^2 - \nu \kappa \\
 &= (0 - 0)^2 - (0 - \frac{1}{r^2})^2 - \frac{\nu}{r} \\
 &= -\frac{1}{r^2} - \frac{\nu}{r}
 \end{aligned}$$

$r \leq 1$ :

$$\begin{aligned}
 u_{\text{outer}} &= \frac{\pi - \pi r^2}{100 - \pi r^2} \\
 u_{\text{inner}} &= 1
 \end{aligned}$$

This leads to following Euler-Lagrange equation:

$$\begin{aligned}
 & (I - u_{\text{outer}})^2 - (I - u_{\text{inner}})^2 - \nu \kappa \\
 &= \left(1 - \frac{\pi - \pi r^2}{100 - \pi r^2}\right)^2 - 0 - \frac{\nu}{r} \\
 &= \frac{100 - \pi r^2 - \pi + \pi r^2}{100 - \pi r^2} - \frac{\nu}{r} \\
 &= \left(\frac{100 - \pi}{100 - \pi r^2}\right)^2 - \frac{\nu}{r}
 \end{aligned}$$

(b)

$$\lim_{r \searrow 1} -\frac{1}{r^2} - \frac{\nu}{r} = -1 - \nu$$
$$\lim_{r \nearrow 1} -\frac{100 - \pi}{100 - \pi r^2} - \frac{\nu}{r} = \frac{100 - \pi}{100 - \pi} - \nu = 1 - \nu$$

As the limits differ the Gateaux derivative at  $r = 1$  is not continuous.

$\nu \leq 1$  is a good choice because it ensures that the curve evolves in the right direction for both cases  $r > 1$  and  $r \leq 1$ .

$r > 1$ :

$$\nu \leq 1 \Rightarrow -\frac{1}{r^2} - \frac{\nu}{r} < 0$$

$r \leq 1$ :

$$\nu \leq 1 \Rightarrow \left( \frac{100 - \pi}{100 - \pi r^2} \right)^2 - \frac{\nu}{r} \geq 0$$