Variational Methods for Computer Vision: Solution Sheet 8

Exercise: December 18, 2015

Part I: Theory

1. The energy before and after the merging are given as:

$$\begin{aligned} E_{before} &= \int_{\Omega_1} (I(x) - u_1)^2 \, \mathrm{d}x + \int_{\Omega_2} (I(x) - u_2)^2 \, \mathrm{d}x + \nu |C_{before}|, \\ E_{after} &= \int_{\Omega_1 \cup \Omega_2} (I(x) - u_{merged})^2 \, \mathrm{d}x + \nu |C_{after}|. \end{aligned} \tag{1}$$

Furthermore, we will make use of the following identites:

$$\begin{aligned} \int_{\Omega} (f(x) - \bar{f})^2 \mathrm{d}x &= \int_{\Omega} f(x)^2 \mathrm{d}x - |\Omega| \bar{f}^2, \\ u_{merged} &= \frac{u_1 A_1 + u_2 A_2}{A_1 + A_2}. \end{aligned} \tag{2}$$

A straightforward calculation yields the desired result:

$$\begin{aligned} \delta E &= E_{after} - E_{before} \\ &= \int_{\Omega_1 \cup \Omega_2} (I(x) - u_{merged})^2 - \int_{\Omega_1} (I(x) - u_1)^2 \, \mathrm{d}x - \int_{\Omega_2} (I(x) - u_2)^2 \, \mathrm{d}x - \nu \delta C \\ &= \int_{\Omega_1 \cup \Omega_2} I(x)^2 \mathrm{d}x - (A_1 + A_2) u_{merged}^2 \\ &\quad - \int_{\Omega_1} I(x)^2 \mathrm{d}x + A_1 u_1^2 - \int_{\Omega_2} I(x)^2 \mathrm{d}x + A_2 u_2^2 - \nu \delta C \\ &= -(A_1 + A_2) \left(\frac{u_1 A_1 + u_2 A_2}{A_1 + A_2} \right)^2 + A_1 u_1^2 + A_2 u_2^2 - \nu \delta C \\ &= \frac{A_1 A_2}{A_1 + A_2} (u_1^2 - u_2^2) - \nu \delta C. \end{aligned} \tag{3}$$