Variational Methods for Computer Vision: Solution Sheet 8

Exercise: January 15, 2015

Part I: Theory

1. Using the property that the delta distribution $\delta(\phi)$ is the derivative of H, we get:

$$|\nabla H(\phi)| = |\delta(\phi) \cdot \nabla \phi(x)| = \delta(\phi)|\nabla \phi(x)|$$

So we can rewrite the functional in the following form:

$$\begin{aligned}
\mathcal{L} &= f_1 \cdot H(\phi) + f_2(1 - H(\phi)) + \nu|\nabla H(\phi)| \\
&= (f_1 - f_2)H(\phi) + \nu\delta(\phi)|\nabla \phi(x)|
\end{aligned}$$

So:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \nabla \phi} &= \frac{\partial}{\partial \nabla \phi}\nu|\nabla \phi|\delta(\phi) \\
&= \nu \cdot \delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|}
\end{aligned}$$

And:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= H'(\phi) \cdot (f_1 - f_2) + \nu\delta'(\phi)|\nabla \phi(x)| \\
&= \delta(\phi) \cdot (f_1 - f_2) + \nu\delta'(\phi)|\nabla \phi(x)|
\end{aligned}$$

Using both results we can compute

$$\begin{aligned}
\frac{\partial E}{\partial \phi} &= \frac{\partial \mathcal{L}}{\partial \phi} - \mathrm{div}\left(\frac{\partial \mathcal{L}}{\partial \nabla \phi}\right) \\
&= \delta(\phi) \cdot (f_1 - f_2) + \nu\delta'(\phi)|\nabla \phi(x)| - \nu\,\mathrm{div}\left(\delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|}\right)
\end{aligned}$$

Further simplifying gives:

$$\begin{aligned}
\mathrm{div}\left(\delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|}\right) =& \frac{\mathrm{d}}{\mathrm{d}x}\left(\delta(\phi) \cdot \frac{\nabla \phi}{|\nabla(\phi)|}\right) + \frac{\mathrm{d}}{\mathrm{d}y}\left(\delta(\phi) \cdot \frac{\nabla \phi}{|\nabla(\phi)|}\right) \\
=& \left(\frac{\mathrm{d}}{\mathrm{d}x}\delta(\phi)\right) \cdot \frac{\nabla \phi}{|\nabla(\phi)|} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\nabla \phi}{|\nabla(\phi)|}\right) \cdot \delta(\phi) \\
&+ \left(\frac{\mathrm{d}}{\mathrm{d}y}\delta(\phi)\right) \cdot \frac{\nabla \phi}{|\nabla(\phi)|} + \left(\frac{\mathrm{d}}{\mathrm{d}y}\frac{\nabla \phi}{|\nabla(\phi)|}\right) \cdot \delta(\phi) \\
=& \delta'(\phi) \cdot \phi_x \cdot \frac{\nabla \phi}{|\nabla(\phi)|} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\nabla \phi}{|\nabla(\phi)|}\right) \cdot \delta(\phi) \\
&+ \delta'(x) \cdot \phi_y \cdot \frac{\nabla \phi}{|\nabla(\phi)|} + \left(\frac{\mathrm{d}}{\mathrm{d}y}\frac{\nabla \phi}{|\nabla(\phi)|}\right) \cdot \delta(\phi) \\
=& \delta(\phi)\,\mathrm{div}\left(\frac{\nabla \phi}{|\nabla(\phi)|}\right) + \delta'(\phi) \cdot \nabla \phi \cdot \frac{\nabla \phi}{|\nabla(\phi)|} \\
=& \delta'(\phi)|\nabla \phi| + \delta(\phi) \cdot \mathrm{div}\left(\frac{\nabla \phi}{|\nabla(\phi)|}\right)
\end{aligned}$$

Finally:

$$\frac{\partial E}{\partial \phi} = \delta(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi)|\nabla \phi(x)| - \nu \delta'(\phi)|\nabla \phi| - \nu \delta(\phi) \cdot \operatorname{div}\left(\frac{\nabla \phi}{|\nabla(\phi)|}\right)$$

$$= \delta \phi \left(f_1 - f_2 - \nu \operatorname{div}\left(\frac{\nabla \phi}{|\nabla(\phi)|}\right)\right)$$

2.

$$E(\phi) = \int g(x)|\nabla H(\phi(x))|\mathrm{d}x$$

$$= \int g(x)\delta(\phi)|\nabla \phi|\mathrm{d}x$$

With $\frac{\partial E}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \operatorname{div}\left(\frac{\partial \mathcal{L}}{\partial \nabla \phi}\right)$ this leads to:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \delta'(\phi)g(x)|\nabla \phi|$$

and

$$\frac{\partial \mathcal{L}}{\partial \nabla \phi} = g(x)\delta(\phi)\frac{\nabla \phi}{|\nabla(\phi)|}$$

Combining both results and further evaluating $\operatorname{div}\left(g(x)\delta(\phi)\frac{\nabla \phi}{|\nabla(\phi)|}\right)$ gives:

$$\frac{\partial E}{\partial \phi} = \frac{\partial L}{\partial \phi} - \operatorname{div}\left(\frac{\partial L}{\partial \nabla \phi}\right)$$

$$= \delta'(\phi)g(x)|\nabla \phi| - \operatorname{div}\left(g(x)\delta(\phi)\frac{\nabla \phi}{|\nabla \phi|}\right)$$

$$= \delta'(\phi)g(x)|\nabla \phi| - \delta'(\phi)g(x)\frac{\nabla \phi \cdot \nabla \phi}{|\nabla \phi|} - \delta(\phi)\operatorname{div}\left(g(x)\frac{\nabla \phi}{|\nabla \phi|}\right)$$

$$= -\delta(\phi)\operatorname{div}\left(g(x)\frac{\nabla \phi}{|\nabla \phi|}\right)$$