# Weekly Exercises 2 

Room: 02.05.014
Tuesday, 03.11.2015, 14:15-15:45
Submission deadline: Tuesday, 03.11.2015, 11:15, Room 02.09.023

## Probability Theory

Exercise 1 ( $\sigma$-Algebra, 3 Points).
a) Let $\mathcal{A} \subset \mathcal{P}(\Omega)$, such that

$$
\begin{array}{r}
A \in \mathcal{A} \Rightarrow \Omega \backslash A \in \mathcal{A} \text { and } \\
A_{1}, A_{2}, \ldots \in \mathcal{A} \Rightarrow \bigcup_{i \in \mathbb{N}} A_{i} \in \mathcal{A} .
\end{array}
$$

Show that

$$
\emptyset \in \mathcal{A} \Leftrightarrow \mathcal{A} \neq \emptyset \Leftrightarrow \Omega \in \mathcal{A} .
$$

b) Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ be a $\sigma$-algebra, show that $\mathcal{A}$ is closed under intersections. I.e.

$$
A_{1}, A_{2} \in \mathcal{A} \Rightarrow A_{1} \cap A_{2} \in \mathcal{A}
$$

Solution. a)

$$
\begin{aligned}
& \emptyset \in \mathcal{A} \\
\Rightarrow & \mathcal{A} \neq \emptyset \\
\Rightarrow & \exists A \in \mathcal{A}: \Omega \backslash A \in \mathcal{A} \\
\Rightarrow & A \cup(\Omega \backslash A)=\Omega \in \mathcal{A} \\
\Rightarrow & \Omega \backslash \Omega=\emptyset \in \mathcal{A} .
\end{aligned}
$$

b) Note that if $A_{1}, A_{2} \in \mathcal{A}$ then $A_{1} \backslash A_{2} \in \mathcal{A}$, since

$$
A_{1} \backslash A_{2}=\Omega \backslash\left(\left(\Omega \backslash A_{1}\right) \cup A_{2}\right)
$$

Now the proposition follows from

$$
A_{1} \cap A_{2}=\left(\left(A_{1} \cup A_{2}\right) \backslash\left(A_{1} \backslash A_{2}\right)\right) \backslash\left(A_{2} \backslash A_{1}\right)
$$

Exercise 2 (Bayes' Rule, 3 Points). Siegfried the ornithologist does a study on the green-speckled swallow. Since he has a huge collection of bird photographs he wants to find all images depicting a green-speckled swallow. Due to it's destinctive features it is an easy task for Eduard, Siegfried's friend and computer vision scientist, to program a green-speckled swallow detector that marks all images containing such a bird. Unfortunately the detector does not work perfectly. If the image contains a green-speckled swallow the detector marks it correctly with a chance of $99.5 \%$. If the image does not contain a green-speckled swallow the detector marks it correctly with a chance of $99.3 \%$. The bird is also very rare: If we randomly draw an image from the collection, there is only an chance of $0.001 \%$ that the image contains a green-speckled swallow.
a) Do a formal modeling of the experiment. How does the discrete probability space look like?
b) What is the probability that a green-speckled swallow is on a given image, if the detector gives a positive answer? What is the probability that a green-speckled swallow is on a given image, if the detector gives a negative answer?

Solution. a) The probability space $(\Omega, \mathcal{A}, P)$ is defined by

$$
\begin{aligned}
& \Omega=\{(s,+),(s,-),(n,+),(n,-)\}, \\
& \mathcal{A}=\{A \subset \Omega\}, \\
& P: \mathcal{A} \rightarrow[0,1],
\end{aligned}
$$

where $s, n,+,-$ have the following meaning:
$s \rightarrow$ image contains a green-speckled sparrow,
$n \rightarrow$ image does not contain a green-speckled sparrow, $+\rightarrow$ detector reports a green-speckled sparrow and
$-\rightarrow$ detector reports no green-speckled sparrow.
Let $X: \Omega \rightarrow\{s, n\},(x, y) \mapsto x$ and $Y: \Omega \rightarrow\{+,-\},(x, y) \mapsto y$, be random variables. From the text we know that $P(Y=+\mid X=s)=0.995, P(Y=-\mid X=$ $n)=0.993$ and $P(X=s)=P(\{(s,+),(s,-)\})=0.00001$.
b)

$$
\begin{aligned}
P(X=n) & =1-P(X=s)=0.99999 \\
P(\{(s,+)\}) & =P(X=s, Y=+)=P(Y=+\mid X=s) \cdot P(X=s)=9.995 \cdot 10^{-6} \\
P(\{(s,-)\}) & =P(\{(s,+),(s,-)\})-P(\{(s,+)\})=5 \cdot 10^{-9} \\
P(\{(n,-)\}) & =P(X=n, Y=-)=P(Y=-\mid X=n) \cdot P(X=n)=0.99299 \\
P(\{(n,+)\}) & =1-P(\{(s,+),(s,-)\})-P(\{(n,-)\})=0.007 \\
P(Y=+) & =P(\{(s,+)\})+P(\{(n,+)\})=7.009995 \cdot 10^{-3} \\
P(Y=-) & =P(\{(s,-)\})+P(\{(n,-)\})=0.99299005 \\
P(X=s \mid Y=+) & =\frac{P(\{(s,+)\})}{P(Y=+)}=0.0014258 \\
P(X=s \mid Y=-) & =\frac{P(\{(s,-)\})}{P(Y=-)}=4.96495 \cdot 10^{-9} .
\end{aligned}
$$

Exercise 3 (Probability, 4 Points). In order to express his gratitude, Siegfried invites Eduard to a pub for a couple of beers. There, they start playing a friendly game of darts.

The dart board is a perfect disk of radius 10 cm . If a dart falls within 1 cm of the center, 100 points are scored. If the dart hits the board between 1 and 3 cm from the center, 50 points are scored, if it is at a distance of 3 to 5 cm 25 points are scored and if it is further away than 5 cm 10 points are scored.

As Siegfried and Eduard are both quite experienced dart players, they hit the dart board every time. Siegfried places the dart uniformly on the board.
a) Define the probability space $(\Omega, \mathcal{A}, P)$ and the random variable $X$.
b) What is the probability that Siegfried scores 100 points on one throw?
c) What is the probability of him scoring 50 points on one throw?
d) Eduard is very focused and thus twice more likely to hit the inner 4 cm part of the board than the outer region. On each region, the dart arrives uniformly. Answer the previous questions now for Eduard's throw.

## Solution.

a)

$$
\begin{aligned}
& \Omega=\left\{(x, y) \in \mathbb{R}^{2} \mid \sqrt{x^{2}+y^{2}} \leq 10\right\}, \\
& \mathcal{A}=\left\{A \subset \Omega \mid \int_{\Omega} \chi_{A}(x) \mathrm{d} x \text { exists. }\right\}, \\
& P: \mathcal{A} \rightarrow[0,1], P(A)=\frac{\int_{\Omega} \chi_{A}(x) \mathrm{d} x}{100 \pi}, \\
& X: \Omega \rightarrow\{10,25,50,100\}, X(x)= \begin{cases}100, & \text { if } 0 \leq\|x\|_{2} \leq 1 \\
50, & \text { if } 1 \leq\|x\|_{2} \leq 3 \\
25, & \text { if } 3 \leq\|x\|_{2} \leq 5 \\
10, & \text { if } 5 \leq\|x\|_{2} \leq 10\end{cases}
\end{aligned}
$$

b)

$$
p(X=100)=p\left(X^{-1}(100)\right)=\frac{\pi}{100 \pi}=0.01
$$

c)

$$
p(X=50)=p\left(X^{-1}(50)\right)=\frac{9 \pi-\pi}{100 \pi}=\frac{8}{100}=0.08
$$

d) Let $A \subset \Omega$ denote the event of the dart hitting the inner 4 cm region. Then

$$
P(A)+P(\Omega \backslash A)=1 \quad \Rightarrow \quad 2 P(\Omega \backslash A)+P(\Omega \backslash A)=1
$$

thus $P(A)=\frac{2}{3}$.
Let $C \subset \Omega$ denote the event of the dart hitting the inner-most region and $D \subset \Omega$ the event of the dart hitting the $1-3 \mathrm{~cm}$ region. Then

$$
\begin{gathered}
P(C)=P(C \mid A) P(A)=\frac{\pi}{16 \pi} \frac{2}{3}=\frac{1}{24} \\
P(D)=P(D \mid A) P(A)=\frac{9 \pi-\pi}{16 \pi} \frac{2}{3}=\frac{1}{3}
\end{gathered}
$$

## Graphical Models

(5 Points)
Exercise 4 (Conditional Independence, 2 Points). Consider four random variables $a, b, c, d$. Prove the following:

$$
a \Perp(b, c) \mid d \quad \text { implies } \quad a \Perp b \mid d .
$$

Solution. Note that

$$
\begin{aligned}
& x \Perp y \mid v \Leftrightarrow p(x \mid y, v)=p(x \mid v) \\
& x \Perp y \mid v \Leftrightarrow p(x, y \mid v)=p(x \mid v) p(y \mid v) .
\end{aligned}
$$

Since $a$ is conditionally independent of $b, c$ given $d$, we have

$$
p(a, b, c \mid d)=p(a \mid d) p(b, c \mid d)
$$

Summing both sides over all realizations of $c$, we have

$$
\begin{aligned}
& \sum_{c} p(a, b, c \mid d)=\sum_{c} p(a \mid d) p(b, c \mid d) \Leftrightarrow \\
& P(a, b \mid d)=p(a \mid d) p(b \mid d)
\end{aligned}
$$

which is $a \Perp b \mid d$.
Exercise 5 (Bayesian Network, 1 Point). Provide the the factorization $p\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right)$ for the following directed graph:


Solution. The joint distribution over all 5 variables is given as:

$$
p\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right)=p\left(x_{1}\right) p\left(x_{2}\right) p\left(x_{3} \mid x_{1}, x_{2}\right) p\left(x_{4} \mid x_{3}\right) p\left(x_{5} \mid x_{3}\right)
$$

Exercise 6 (MRF, 2 Points). Let $G$ be a factor graph given by a Markov random field consisting of $N^{2}$ binary variables, representing the pixels of a $N \times N$ image. For each pixel there is a unary potential, and there are pairwise potentials according to the 4-connected neighbourhood.
a) Draw the factor graph for $N=3$.
b) How many factors (of each type) are there, depending on $N$ ?

## Solution.

a) The factor graph for $N=3$ is given as the following:

b) Number of factors: $2(N-1)^{2}+2(N-1)+N^{2}$

