

Weekly Exercises 4

Room: 02.09.023

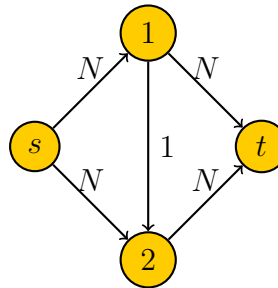
Tuesday, 17.11.2015, 14:15-15:45

Submission deadline: Tuesday, 17.11.2015, 11:15 , Room 02.09.023

Linear Programming

(15 Points)

Exercise 1 (5 Points). Consider the following network $G = (V, E, c, s, t)$:



The numbers on the edges indicate the edge capacities and $N \in \mathbb{N}, N > 1$.

- a) Write down the standard form of the corresponding max-flow problem's linear program. Get rid of variable z in order to transform the constraint matrix into a matrix of maximal rank.

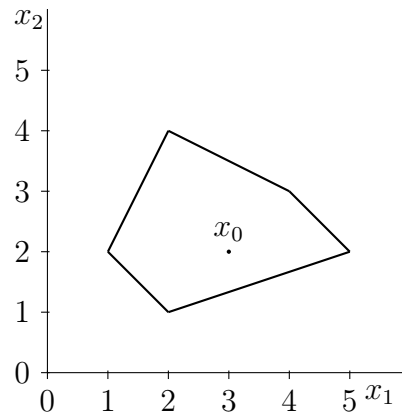
Hint: You can remove two rows of the divergence matrix and move the divergence of node s or t into the objective.

- b) Solve the linear program with the help of the simplex algorithm. Write down the simplex tableau and draw the corresponding flow at each iteration.

Exercise 2 (4 Points). Consider the following linear program:

$$\begin{aligned} \max_{x \in \mathbb{R}^2} \quad & 1x_1 + 4x_2 \\ \text{s.t.} \quad & \begin{pmatrix} -1 & -1 \\ 1 & -3 \\ 1 & 1 \\ 1 & 2 \\ -2 & 1 \end{pmatrix} x \leq \begin{pmatrix} -3 \\ -1 \\ 7 \\ 10 \\ 0 \end{pmatrix} \\ & x \geq 0. \end{aligned}$$

- Bring the linear program into standard form.
- Compute a feasible basic solution of the LP using the method described in the lecture (*Fundamental Theorem of LP*) by starting at the feasible point $x_0 = \left(\frac{3}{2}\right)$.
- Find an optimal basic solution. Is the optimal basic solution unique?
- Can you find the optimum by drawing into the figure below?



Exercise 3 (2 Points). Let $n \geq 2$, $0 < \varepsilon < \frac{1}{2}$. Consider the following linear program from the lecture:

$$\begin{aligned} & \max_{x \in \mathbb{R}^n} x_n, \\ \text{subject to: } & 0 \leq x_1 \leq 1, \\ & \varepsilon x_i \leq x_{i+1} \leq 1 - \varepsilon x_i, \quad \text{for all } i = 1, \dots, n-1. \end{aligned}$$

- Draw the feasible set for $n = 2$, $\varepsilon = \frac{1}{3}$.
- For $n = 3$, $\varepsilon = \frac{1}{3}$, derive an order of basic feasible solutions with strictly increasing energy where each basic feasible solution is visited once.

Exercise 4 (4 Points). Let $A \in \{-1, 0, 1\}^{m \times n}$ be a matrix with two different non-zero entries in each row.

- Show that A is totally unimodular.
Hint: Perform induction over the size of square submatrices $B_k \in \{-1, 0, 1\}^{k \times k}$ to show that $\det B_k \in \{-1, 0, 1\}$.
- Use the previous result to show that the gradient $\text{Grad} : \mathbb{R}^{|V|} \rightarrow \mathbb{R}^{|E|}$ is totally unimodular.
- Argue that the divergence $\text{div} : \mathbb{R}^{|E|} \rightarrow \mathbb{R}^{|V|}$ is also totally unimodular.

Programming (0 Points)

Complete the programming assignment from last week.