Combinatorial Optimization in Computer Vision
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Weekly Exercises 4

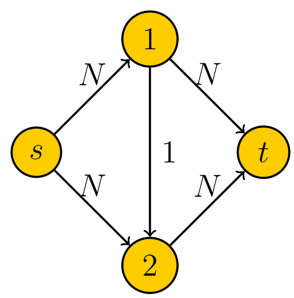
Room: 02.09.023

Tuesday, 17.11.2015, 14:15-15:45

Submission deadline: Tuesday, 17.11.2015, 11:15 , Room 02.09.023

Linear Programming (15 Points)

Exercise 1 (5 Points)**.** Consider the following network $G = (V, E, c, s, t)$:



The numbers on the edges indicate the edge capacities and $N \in \mathbb{N}, N > 1$.

a) Write down the standard form of the corresponding max-flow problem's linear program. Get rid of variable z in order to transform the constraint matrix into a matrix of maximal rank.

   Hint: You can remove two rows of the divergence matrix and move the divergence of node s or t into the objective.

b) Solve the linear program with the help of the simplex algorithm. Write down the simplex tableau and draw the corresponding flow at each iteration.

Exercise 2 (4 Points)**.** Consider the following linear program:

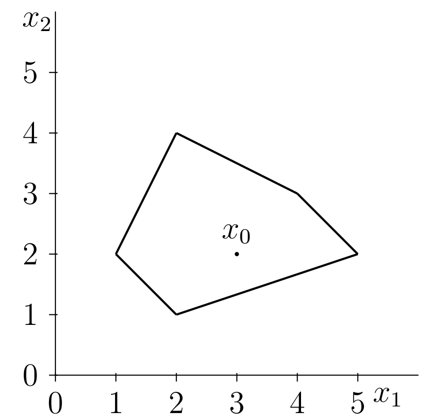
$$\begin{aligned}
\max_{x \in \mathbb{R}^2} \quad & 1x_1 + 4x_2 \\
\text{s.t.} \quad & \begin{pmatrix} -1 & -1 \\ 1 & -3 \\ 1 & 1 \\ 1 & 2 \\ -2 & 1 \end{pmatrix} x \leq \begin{pmatrix} -3 \\ -1 \\ 7 \\ 10 \\ 0 \end{pmatrix} \\
& x \geq 0.
\end{aligned}$$

a) Bring the linear program into standard form.

b) Compute a feasible basic solution of the LP using the method described in the lecture (*Fundamental Theorem of LP*) by starting at the feasible point $x_0 = \binom{3}{2}$.

c) Find an optimal basic solution. Is the optimal basic solution unique?

d) Can you find the optimum by drawing into the figure below?



Exercise 3 (2 Points)**.** Let $n \geq 2$, $0 < \varepsilon < \frac{1}{2}$. Consider the following linear program from the lecture:

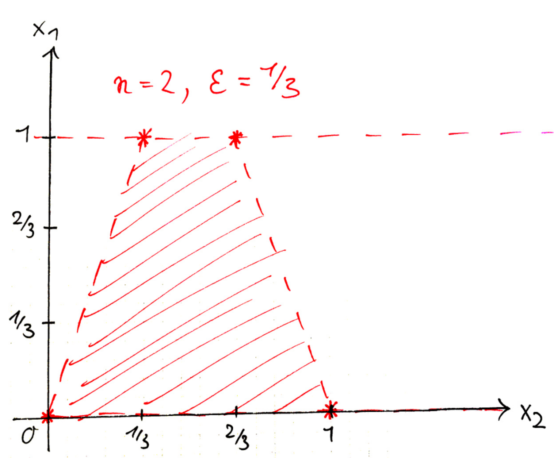
$$\begin{aligned} &\max_{x \in \mathbb{R}^n} \; x_n, \\ \text{subject to:} \quad & 0 \leq x_1 \leq 1, \\ & \varepsilon x_i \leq x_{i+1} \leq 1 - \varepsilon x_i, \qquad \text{for all } i = 1, \dots, n-1. \end{aligned}$$

a) Draw the feasible set for $n = 2$, $\varepsilon = \frac{1}{3}$.

b) For $n = 3$, $\varepsilon = \frac{1}{3}$, derive an order of basic feasible solutions with strictly increasing energy where each basic feasible solution is visited once.

Solution.

a)



b) There are eight vertices $(v_1, \dots, v_8)$ which are lying on a "cube". The last coordinate denotes the energy:

$$\begin{aligned}
v_1 &= (0,0,0) \simeq (0,0,0),\\
v_2 &= (0,0,1) \simeq (0,0,1),\\
v_3 &= \left(0,1,\frac{3}{9}\right) \simeq (0,1,0),\\
v_4 &= \left(0,1,\frac{6}{9}\right) \simeq (0,1,1),\\
v_5 &= \left(1,\frac{1}{3},\frac{1}{9}\right) \simeq (1,0,0),\\
v_6 &= \left(1,\frac{1}{3},\frac{8}{9}\right) \simeq (1,0,1),\\
v_7 &= \left(1,\frac{2}{3},\frac{2}{9}\right) \simeq (1,1,0),\\
v_8 &= \left(1,\frac{2}{3},\frac{7}{9}\right) \simeq (1,1,1).
\end{aligned}$$

The path $(v_1, v_5, v_7, v_3, v_4, v_8, v_6)$ leads to an increasing energy. This path exists, since only 1 coordinate is changing in each step and thus is a valid path on the cube.

Exercise 4 (4 Points)**.** Let $A \in \{-1,0,1\}^{m\times n}$ be a matrix with two different non-zero entries in each row.

a) Show that A is totally unimodular.
Hint: Perform induction over the size of square submatrices $B_k \in \{-1,0,1\}^{k\times k}$ to show that $\det B_k \in \{-1,0,1\}$.

b) Use the previous result to show that the gradient $\mathrm{Grad} : \mathbb{R}^{|V|} \to \mathbb{R}^{|E|}$ is totally unimodular.

c) Argue that the divergence $\mathrm{div} : \mathbb{R}^{|E|} \to \mathbb{R}^{|V|}$ is also totally unimodular.

Solution.

a) Let $B \in \mathbb{R}^{k\times k}$ be a square submatrix of A. We prove that $\det B \in \{-1,0,1\}$ using induction over k. The base case $k = 1$ immediately follows from the definition of A. Now let $k > 1$.

- If B has a complete zero row, then $\det B = 0$.
- If B has a row with exactly one non-zero element, then

$$\det B = \pm \det \begin{pmatrix} \pm 1 & 0 \\ b & B' \end{pmatrix} = \pm \det B',$$

  and $\det B' \in \{-1,0,1\}$ by induction hypothesis

- B has in every row exactly one $+1$ and one -1. Adding up all the columns yields the zero vector. Since adding two columns does not change the determinant, we have $\det B = 0$.

b) As the gradient is precisely a matrix with exactly one $+1$ and one -1 in each row, the claim follows from a).

c) Since $\operatorname{div} = -\operatorname{Grad}^{\mathsf{T}}$ and the transpose of a totally unimodular matrix is totally unimodular again, and as $\det A = \pm \det -A$ we know that the divergence is also totally unimodular.

Programming (0 Points)

Complete the programming assignment from last week.