Computer Vision Group
Institut für Informatik
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Weekly Exercises 7

Room: 02.09.023
Tuesday, 09.12.2015, 14:15-15:45
Submission deadline: Tuesday, 09.12.2015, 11:15 , Room 02.09.023

Convexity (2 Points)

Exercise 1 (2 Points)**.** Show that the following functions are convex:

a) $f : \mathbb{R} \to \mathbb{R},\ x \mapsto |x|^p,\ p \in [1, \infty)$.

b) $f : \mathbb{R} \to \mathbb{R},\ x \mapsto \log(1 + \exp(x))$.

c) $f : \mathbb{R}_{\geq 0} \to \mathbb{R},\ x \mapsto x \log(x)$.

d) $f : \mathbb{R} \to \mathbb{R},\ x \mapsto \max(0, x)$.

Solution.

a) We have $f(x) = g(h(x))$, with $g : \mathbb{R}_{\geq 0} \to \mathbb{R}, x \mapsto x^p$ and $h(x) = |x|$. Since g is increasing and convex on $\mathbb{R}_{\geq 0}$ and h is convex, their composition is convex as well.

b) $f'(x) = \frac{\exp(x)}{1+\exp(x)}$, $f''(x) = \frac{\exp(x)(1+\exp(x))-\exp(x)\exp(x)}{(1+\exp(x))^2} = \frac{\exp(x)}{(1+\exp(x))^2} \geq 0$.

c) $f'(x) = x\frac{1}{x} + \log(x) = \log(x)$, $f''(x) = \frac{1}{x} \geq 0$ for $x \geq 0$.

d) We want to prove that

$$\max(0, \alpha x + (1-\alpha)y) \leq \alpha \max(0, x) + (1-\alpha)\max(0, y).$$

Clearly it holds

$$0 \leq \alpha \max(0, x) + (1-\alpha)\max(0, y),$$

and also

$$\alpha x + (1-\alpha)y \leq \alpha \max(0, x) + (1-\alpha)\max(0, y).$$

Hence the claim follows.

Multi-Label Problems and Submodularity (8 Points)

Let $\mathcal{L} = \{1, \ldots, \ell\} \subset \mathbb{N}, \ell \in \mathbb{N}$, be a totally ordered label set and $n \in \mathbb{N}$. The total order on $\mathcal{L}$ induces a partial order on $\mathcal{L}^n$. It is easy to check that $\mathcal{L}^n$ is a distributive lattice, where meet $\wedge$ and join $\vee$ are the component-wise minimum and maximum, respectively. I.e.

$$\begin{aligned}\big(x \wedge y\big)_i &= \big(\min\{x_i, y_i\}\big)_i \text{ and} \\ \big(x \vee y\big)_i &= \big(\max\{x_i, y_i\}\big)_i\end{aligned} \tag{1}$$

for any $x, y \in \mathcal{L}^n$.

Definition. A function $c : \mathcal{L}^n \to \mathbb{R}$ is called *submodular* if for any $x, y \in \mathcal{L}^n$ it holds

$$c(x \wedge y) + c(x \vee y) \leq c(x) + c(y). \tag{2}$$

Exercise 2 (4 Points)**.**

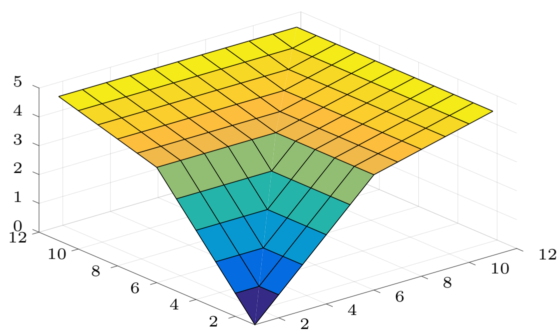
a) Let $g : \mathbb{R} \to \mathbb{R}$ be a convex function. Prove that the function $c : \mathcal{L} \times \mathcal{L} \to \mathbb{R}, c(x, y) = g(|x - y|)$, is submodular.

b) Let $f_i : \mathcal{L} \to \mathbb{R}$ be some functions and let $c_{i,j} : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ be some pair-wise submodular functions. Prove that the energy $E : \mathcal{L}^n \to \mathbb{R}$, given by

$$E(x) = \sum_i f_i(x_i) + \sum_{i,j} c_{i,j}(x_i, x_j), \tag{3}$$

is submodular.

c) Let $a, b, c \in \mathbb{R}_{\geq 0}$ be some constants. Show that $c : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$, where $c(x, y) = \min\{a \max\{x, y\}, b \max\{x, y\} + c\}$, is submodular but not necessarily convex.



Exercise 3 (4 Points)**.** Let $E : \mathbb{B}^n \to \mathbb{R}$ be a quadratic submodular energy and $x \in \mathbb{B}^n$. Further, let $x^1 \in \mathbb{B}^n$ be the result of the 1-expansion starting from x and let $x^0 \in \mathbb{B}^n$ be the result of the 0-expansion starting from x^1.

a) Show that there is a $x^* \in \arg\min_z E(z)$ such that $x^* \leq x^1$.

b) Show that $x^0 \in \arg\min_z E(z)$.

Solution. First let us recall the definition of the α-Expansion starting from z. Given a current labeling $z \in \mathcal{L}^n$ and label $\alpha \in \mathcal{L}$ we obtain a minimizer $y^* \in \mathbb{B}^n$ of the following energy:

$$E^{z,\alpha}(y) = \sum_{i=1}^{n} f_i^{z,\alpha}(y_i) + \sum_{i=1}^{n} \sum_{j \in \mathcal{N}(i)} f_{ij}^{z,\alpha}(y_i, y_j),$$

where

$$\begin{aligned} f_i^{z,\alpha}(0) &= f_i(z_i), & f_i^{z,\alpha}(1) &= f_i(\alpha), \\ f_{ij}^{z,\alpha}(0,0) &= f_{ij}(z_i,z_j), & f_{ij}^{z,\alpha}(0,1) &= f_{ij}(z_i,\alpha), \\ f_{ij}^{z,\alpha}(1,0) &= f_{ij}(\alpha,z_j), & f_{ij}^{z,\alpha}(1,1) &= f_{ij}(\alpha,\alpha). \end{aligned}$$

Subsequently, we set

$$z_i^{\text{new}} = \begin{cases} \alpha, & \text{if } y_i = 1, \\ z_i, & \text{otherwise.} \end{cases}$$

Clearly it holds that $E^{z,\alpha}(y^*) = E(z^{\text{new}})$ by definition. Since y^* is a global minimizer, $z^{\text{new}} \in \mathcal{L}^n$ is the optimal solution that can be obtained by setting some variables to the label α.

a) Let us define the sets

$$\begin{aligned} A &= \{i \mid x_i^1 = 1\}, & (4) \\ B &= \{i \mid x_i^* = 1\}, & (5) \\ C &= \{i \mid x_i^0 = 1\}, & (6) \\ X &= \{i \mid x_i = 1\}, & (7) \\ S &= A \cap B. & (8) \end{aligned}$$

In the following we identifity sets with binary vectors in $\mathbb{B}^n$ as defined above, i.e., we interchangingly use $E(A)$ for $E(x^1)$.

Since E is submodular, we have

$$E(A \cup B) \leq E(A) + \underbrace{E(B) - E(S)}_{\leq 0} \leq E(A). \tag{9}$$

Since A is obtained by the 1-expansion starting from X, we have that $X \subset A$ and that A is the set with the overall minimum energy containing X, i.e. we have that:

$$E(A) \leq E(Y),\ \forall Y \supset X.$$

Since A is the set with the overall minimum energy containing X and clearly $A \cup B$ also contains X and $E(A \cup B) \leq E(A)$ we have that

$$E(A) = E(A \cup B).$$

Hence, it follows from (9) that

$$E(S) \leq E(B),$$

which in turn shows that $S \in \operatorname{argmin}_z E(z)$ since B is the global minimizer of E. Since $S \subset A$ we have that $x^* \leq x^1$ for some global minimizer $x^* \in \operatorname{argmin}_z E(z)$.

b) According to the definition of the 0-expansion starting from A, C is the set with the minimal energy having the property $\bar{A} \subset \bar{C}$, i.e.

$$E(C) \leq E(Y),\ \forall \bar{Y} \supset \bar{A}.$$

Since from a) we have $B \subset A$ we have that $\bar{B} \supset \bar{A}$ and hence $E(C) \leq E(B)$. Hence C is a global minimizer, i.e. $x^0 \in \operatorname{argmin}_z E(z)$.

Programming (10 Points)

Exercise 4 (Image Denoising, 10 Points)**.** Given a noisy input image $I \in \mathcal{L}^n$ consisting of n pixels and $\mathcal{L} = \{1, \ldots, \ell\}$ intensities, the goal of this exercise is to compute a denoised version $x \in \mathcal{L}^n$ of I (see Figure 1).

To this end, we want to minimize the following energy:

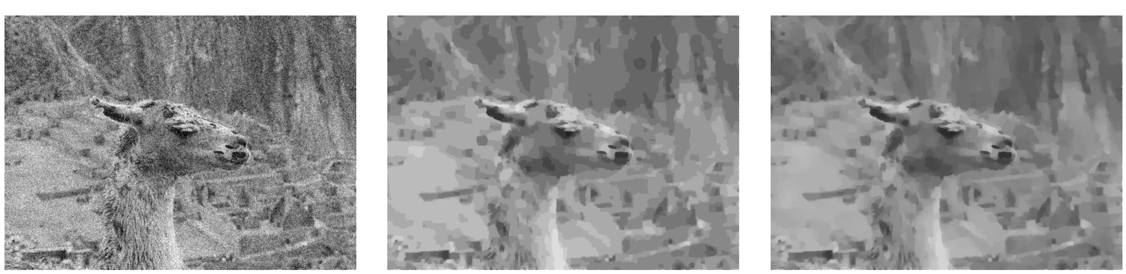
$$E(x) = \sum_{i=1}^{n} f_i(x_i) + \sum_{i=1}^{n} \sum_{j \in \mathcal{N}(i)} f_{ij}(x_i, x_j). \tag{10}$$

The unary potentials $f_i : \mathcal{L} \to \mathbb{R}$ are chosen accordingly to the statistical noise model. In this exercise we assume Gaussian noise, i.e. $f_i(x_i) = \frac{1}{2}(x_i - I_i)^2$. The pairwise potentials $f_{ij} : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ are used to model the prior knowledge about the input image. Here we pick $f_{ij}(x_i, x_j) = \lambda |x_i - x_j|^p$ for some parameters $\lambda > 0$, $p \geq 1$. Chose a simple 4-connected neighbourhood for $\mathcal{N}$.

Download `07_supp.zip` from the lecture website and use `denoising.cpp` as a starting point which loads an image, converts it into grayscale and adds Gaussian noise with standard deviation σ. Extend the program in the following ways:

a) Formulate the multilabel energy minimization problem (10) as a pseudo-Boolean optimization problem using the construction presented in the lecture. Use Kolmogorov's code[1] for graph cuts to find a global minimizer of the submodular pseudo-Boolean energy equivalent to (10). The graph construction of the dataterm is already provided in the code `denoising.cpp` and your task is to add the edges for the regularizer.

b) Formulate the energy minimization problem (10) as a multilabel optimization problem in OpenGM and apply the α-Expansion and α-β-Swap algorithms (these are already implemented in OpenGM). Compare the results from the different algorithms to the globally optimal result of exercise a).



Noisy llama | $\ell = 16$ | $\ell = 128$

Figure 1: Examplary denoising results for $p = 1$ and $\lambda = 0.25$ and different number of labels.

[1] http://pub.ist.ac.at/~vnk/software.html