Combinatorial Optimization in Computer Vision
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Weekly Exercises 10

Room: 02.09.023

Tuesday, 19.01.2016, 14:15-15:45

Submission deadline: Tuesday, 19.01.2016, 11:15 , Room 02.09.023

Fast Primal-Dual (5 Points)

Exercise 1 (Multilabeling ILP, 3 Points)**.** Given a multilabel problem with $n = 2$ pixels, connected by an edge set $\mathcal{E}$ forming a complete graph and $m = 3$ labels ($\mathcal{L} = \{1, 2, 3\}$), explicitly write out the linear programming formulation from the lecture

$$\min_x \; \langle c, x \rangle$$
$$Ax = b, x \geq 0.$$

in matrix-vector notation, i.e. state the following quantities element by element:

- $A \in \mathbb{R}^{(n+2|\mathcal{E}|m)\times(nm+|\mathcal{E}|m^2)}$,
- $b \in \mathbb{R}^{n+2|\mathcal{E}|m}$,
- $c \in \mathbb{R}^{nm+|\mathcal{E}|m^2}$.

Assume that $d(\cdot, \cdot)$ is the Potts metric.

Exercise 2 (Complementary slackness, 2 Points)**.** Let (x, y) be a pair of integral primal and dual feasible solutions to the linear programming relaxation of the multilabel problem:

$$\min_x \; \langle c, x \rangle \qquad\qquad \max_y \; \langle b, y \rangle$$
$$Ax = b, x \geq 0. \qquad\qquad A^T y \leq c.$$

If (x, y) satisfy the relaxed primal complementary slackness conditions

$$\forall x_j > 0 \Rightarrow \sum_{i=1}^{m} a_{ij} y_i \geq c_j / \varepsilon_j,$$

show that then x is an ε-approximation to the optimal integral solution x^* with $\varepsilon = \max_j \varepsilon_j$.

Branch and Bound (2 Points)

Exercise 3 (Lower bound, 2 Points)**.** For a finite set Ω, consider the following segmentation energy function $E : \mathbb{B}^n \to \mathbb{R}$:

$$E(x) = \min_{\omega \in \Omega} C(\omega) + \sum_{i=1}^{n} f_i(\omega) x_i + \sum_{i=1}^{n} b_i(\omega)(1 - x_i) + \sum_{i=1}^{n} \sum_{j \in \mathcal{N}(i)} w_{ij}(\omega)|x_i - x_j|, \quad (1)$$

with $C : \Omega \to \mathbb{R}$, $f_i : \Omega \to \mathbb{R}$, $b_i : \Omega \to \mathbb{R}$, $w_{ij} : \Omega \to \mathbb{R}$. Prove the following lower bound:

$$\begin{aligned} E(x) \geq \left(\min_{\omega \in \Omega} C(\omega) \right) + \sum_{i=1}^{n} \left(\min_{\omega \in \Omega} f_i(\omega) \right) x_i + \sum_{i=1}^{n} \left(\min_{\omega \in \Omega} b_i(\omega) \right) (1 - x_i) \\ + \sum_{i=1}^{n} \sum_{j \in \mathcal{N}(i)} \left(\min_{\omega \in \Omega} w_{ij}(\omega) \right) |x_i - x_j| =: \ell(x, \Omega). \end{aligned} \quad (2)$$

Remark: This shows that $E^* = \min_x E(x) \geq \min_x \ell(x, \Omega) = L(\Omega)$ and $L(\Omega)$ is a lower bound for the global optimum. Note that the lower bound $L(\Omega)$ fulfills three important properties which make it applicable for branch and bound optimization methods:

1. **Monotonicity:** $\Omega_1 \subset \Omega_2 \Rightarrow L(\Omega_1) \geq L(\Omega_2)$.
2. **Computability:** Evaluating $L(\Omega)$ for some given Ω corresponds to minimizing a submodular quadratic pseudo-Boolean function.
3. **Tightness:** For $|\Omega| = 1$, i.e. $\Omega = \{\omega\}$ we have $L(\{\omega\}) = \min_x E(x)$.

Programming (15 Points)

Exercise 4 (Branch-and-Mincut[1], 15 Points)**.** In this exercise we apply the branch and bound method from the lecture to find a *global minimizer* of a discrete version of the celebrated Chan-Vese[2] segmentation energy function:

$$
\begin{aligned}
E(x, \{c_f, c_b\}) = \mu \sum_{i=1}^{n} \sum_{j \in \mathcal{N}(i)} |x_i - x_j| \\
+ \sum_{i=1}^{n} \left(\nu + \lambda_1 (I_i - c_f)^2\right) x_i + \sum_{i=1}^{n} \lambda_2 (I_i - c_b)^2 (1 - x_i).
\end{aligned} \tag{3}
$$

Here I denotes a gray-scale input image with n pixels, i.e. at every pixel $1 \leq i \leq n$ we have $I_i \in [0, 255]$. The variable $\omega = (c_f, c_b) \in \Omega = [0, 255]^2$ denotes the mean intensity of foreground respectively the background of the segmentation $x \in \mathbb{B}^n$.

a) Find an approximate solution of (13) by alternatingly optimizing over x and ω:

$$
\begin{aligned}
x^{k+1} &\in \operatorname{argmin}_x \; E(x, \omega^k), \\
\omega^{k+1} &= \operatorname{argmin}_\omega \; E(x^{k+1}, \omega).
\end{aligned}
$$

The optimization problem in x is a 2-region segmentation problem, so reuse your code from the previous exercises. The problem in ω has a simple closed form solution. Use `chanvese_alternating.cpp` from `10_supp.zip` as a start.

b) Compute a global minimizer of (13) using the branch and bound best-first tree search. The search space Ω is the rectangle $[0, 255]^2$. In your implementation, you can keep a sorted queue of rectangles Ω_i, and every iteration remove the rectangle with the smallest lower bound and split it into two smaller rectangles along the longest edge. As a lower bound on (13) use the bound (8) dervied in the theoretical exercise. You can use `chanvese_global.cpp` as a starting point.

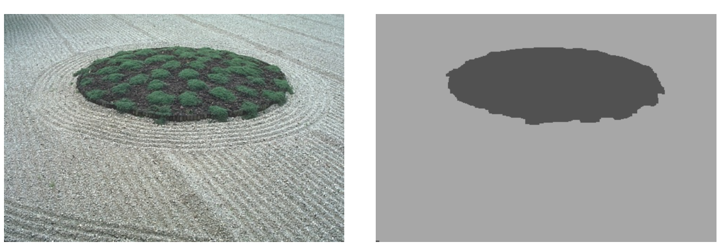

Figure 1: The figure shows the input image and a global minimizer of (13) for parameters $\lambda_1 = \lambda_2 = 0.0001$, $\mu = 1$, $\nu = 0.1$. The optimal foreground and background colors were found as $c_f^* = 81$ and $c_b^* = 167$.

[1] V. Lempitsky, A. Blake, C. Rother, Image Segmentation by Branch-and-Mincut, ECCV 2008

[2] T. Chan, L. Vese: Active contours without edges. Trans. Image Process., 10(2), 2001.