Combinatorial Optimization in Computer Vision
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Weekly Exercises 11

Room: 02.09.023

Tuesday, 19.01.2016, 14:15-15:45

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Fast Trust-Region (6 Points)

Exercise 1 (Bhattacharyya Distance, 3 Points)**.** Let $K = \{1, \dots, k\}, k \in \mathbb{N}$ and let $\mathcal{D} = \{p : K \to \mathbb{R}_{\geq 0} | \sum_i p(i) = 1\}$ be the set of all probability distributions on K. Define the *Bhattacharyya distance* $D : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ by

$$D(p,q) = -\log \left(\sum_i \sqrt{p(i)q(i)} \right).$$

1. Show that the Bhattacharyya distance D is symmetric and non-negative.
2. Show that $D(p,q) = 0 \Leftrightarrow p = q$.
3. Give an example where the triangle inequality does not hold for D.

Solution.

1. Symmetry follows directly from the definition.

$$\begin{aligned} \sum_i \sqrt{p(i)q(i)} &= \sum_i p(i) \sqrt{\frac{q(i)}{p(i)}} \\ &= -\sum_i p(i) \left(-\sqrt{\frac{q(i)}{p(i)}}\right) \\ &\leq \sqrt{\sum_i p(i) \frac{q(i)}{p(i)}} = \sqrt{\sum_i q(i)} = 1. \end{aligned}$$

   The inequality comes from Jensen's inequality $f(\sum_i \lambda_i x_i) \leq \sum_i \lambda_i f(x_i)$, where $f(\cdot) = -\sqrt{\cdot}$ is a convex function. It directly follows that $D(p,q) \geq 0$.

2. If $p = q$ then

$$\sum_i \sqrt{p(i)q(i)} = \sum_i \sqrt{p(i)^2} = \sum_i p(i) = 1.$$

   Therefore $D(p,q) = 0$.

If $D(p,q)=0$ then $\sum_i \sqrt{p(i)q(i)} = 1$. Also

$$\begin{aligned}\sum_i \sqrt{p(i)q(i)} &= \sum_i \sqrt{p(i)}\sqrt{q(i)} \\ &= \langle \sqrt{p}, \sqrt{q} \rangle \\ &\leq \sqrt{\langle \sqrt{p}, \sqrt{p} \rangle}\sqrt{\langle \sqrt{q}, \sqrt{q} \rangle} \\ &= \sqrt{\sum_i p(i)}\sqrt{\sum_i q(i)} = 1.\end{aligned}$$

The inequality is the Cauchy-Schwarz inequality and equality holds if and only if p, q are linear dependent.

3. Let $k = 2$, $x(1) = 0.1, x(2) = 0.9, y(1) = 0.9, y(2) = 0.1, z(1) = z(2) = 0.5$. We can calculate

$$\begin{aligned}D(x,y) &= -\log 2\sqrt{0.09} \approx 0.2218 \\ D(x,z) = D(z,y) &= -\log(\sqrt{0.05} + \sqrt{0.45}) \approx 0.0506.\end{aligned}$$

Therefore $D(x,y) > D(x,z) + D(z,y)$.

For the next exercise we need the *isoperimetric inequality*:

Theorem (Isoperimetric Inequality)**.** Consider a closed curve $\alpha \subset \mathbb{R}^2$. Let $\mathcal{L}$ be the length of α and let $\mathcal{A}$ be the area of the region enclosed by α. It holds:

$$4\pi\mathcal{A} \leq \mathcal{L}^2.$$

Equality holds if and only if α is a circle.

Exercise 2 (3 Points)**.** Consider a minimizer $S^* \subset \mathbb{R}^2$ of the minimization problem

$$\min_{S\subset\mathbb{R}^2} \left(\int_S 1 dx - V_0 \right)^2 + \mathcal{L}(S),$$

where $V_0 \in \mathbb{R}_{>0}$ and $\mathcal{L}(S)$ is the length of the boundary of S.

1. Show that S^* is a disk.

2. Show that the radius r^* of the disk S^* satisfies

$$r^* = \arg\min_{r\geq 0} \left(\pi r^2 - V_0\right)^2 + 2\pi r.$$

3. Find the optimal radius for $V_0 = 1000$. *Hint:* Compute the derivative and use your favourite software (e.g. Matlab or Wolfram α) to find the roots of the resulting third-degree polynomial.

Solution. 1. Assume S^* is not a disk and let T be a disk with the same area $\mathcal{A}(T) = \mathcal{A}(S^*)$. Then

$$\frac{\mathcal{L}(T)^2}{4\pi} = \mathcal{A}(T) = \mathcal{A}(S^*) < \frac{\mathcal{L}(S^*)^2}{4\pi}.$$

I.e. $\mathcal{L}(T) < \mathcal{L}(S^2)$ and $(\int_{S^*} 1dx - V_0)^2 + \mathcal{L}(S^*) = (\mathcal{A}(S^*) - V_0)^2 + \mathcal{L}(S^*) > (\mathcal{A}(T) - V_0)^2 + \mathcal{L}(T)$ contradicting the optimality of S^*.

2. We can reduce the search space to the space of all disks $S(r)$ parameterized by radius r. Then $\mathcal{A}(S(r)) = \pi r^2$ and $\mathcal{L}(S(r)) = 2\pi r$.

3. The derivative of the optimization function is $4\pi^2 r^3 - 4\pi V_0 r + 2\pi$. Setting $V_0 = 1000$ it has real roots at $-17.8415, 0.0005, 17.8415$. The minimum (s.t. $r \geq 0$) is at $r = 17.8415$ and has the value 112.099.