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Linear Programming Simplex Method Dual LP Graph Cut as LP

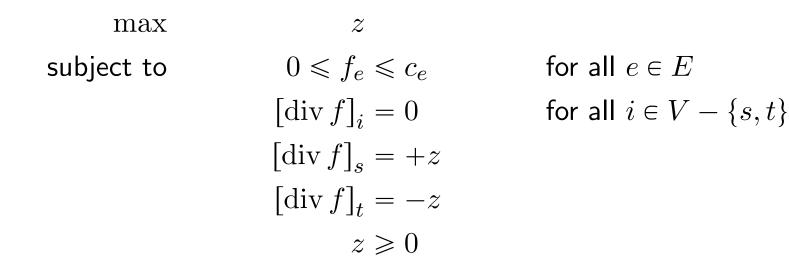
7. Linear Programming

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Linear Programming



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	Linear Programming	Simplex Method	Dual LP	Graph Cut as LP	

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Here, div: $\mathbb{R}^{|E|} \to \mathbb{R}^{|V|}$ is a linear mapping that maps information from the edges to information on the vertices.

Note that the **objective function** and the **constraints** are all linear.



A Linear Program (LP) is an optimization problem of a linear function with respect to linear constraints, i.e.,

$$\begin{cases} \min \\ \max \end{cases}_{x \in \mathbb{R}^n} \langle c, x \rangle \\ \text{subject to} \langle a_i, x \rangle \begin{cases} \leqslant \\ = \\ \geqslant \end{cases} b_i \end{cases}$$

for all
$$i = 1, \ldots, m$$



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If we use the following partial ordering on \mathbb{R}^n :

 $x \leqslant y \qquad \Leftrightarrow \qquad x_k \leqslant y_k \qquad \text{for all } k = 1, \dots, n$

we can simplify the notation of LPs.



An LP is in **canonical form** if it is of the form

 $\max_{x \in \mathbb{R}^n} \langle c, x \rangle$
subject to $Ax \leq b$
 $x \geq 0$

for a constraint matrix $A \in \mathbb{R}^{m \times n}$, a constraint vector $b \in \mathbb{R}^m$ and a cost vector $c \in \mathbb{R}^n$. We have *n* variables and *m* constraints.



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An LP is in **standard form** if it is of the form

 $\max_{x \in \mathbb{R}^n} \langle c, x \rangle$
subject to Ax = b
 $x \ge 0$



Every minimization problem becomes an equivalent maximization problem by replacing c with -c.



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Note that the following equivalences can transform the constraints of an LP in purely " \leq " or "=" constraints:

$$\begin{array}{ll} \langle a_i, x \rangle \geqslant b_i & \Leftrightarrow & \langle -a_i, x \rangle \leqslant -b_i \\ \langle a_i, x \rangle = b_i & \Leftrightarrow & \langle +a_i, x \rangle \leqslant +b_i, \\ \langle -a_i, x \rangle \leqslant -b_i & \langle -a_i, x \rangle \leqslant -b_i \\ \langle a_i, x \rangle \leqslant b_i & \Leftrightarrow & \langle a_i, x \rangle + s_i = b_i \end{array}$$

The extra variable in the last equivalence is called **slack variable** $s_i \ge 0$.



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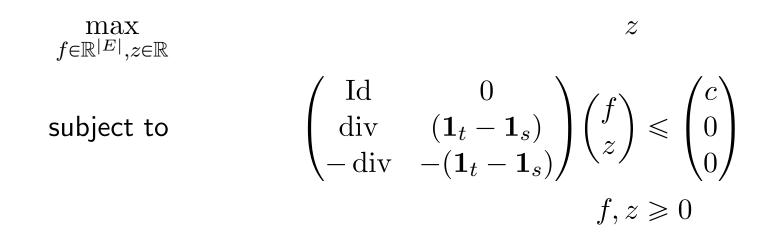
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The extra variable in the last equivalence is called **slack variable** $s_i \ge 0$.

If a variable x_i is not constrained $(x_i \ge 0)$, one can use two constrained variables $x_i^+, x_i^- \ge 0$ and replace each occurrence of x_i with $x_i^+ - x_i^-$.



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$$\begin{aligned} \max_{f \in \mathbb{R}^{|E|}, z \in \mathbb{R}} & z \\ \text{subject to} & \begin{pmatrix} \text{Id} & 0 \\ \text{div} & (\mathbf{1}_t - \mathbf{1}_s) \\ -\text{div} & -(\mathbf{1}_t - \mathbf{1}_s) \end{pmatrix} \begin{pmatrix} f \\ z \end{pmatrix} \leqslant \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} \\ f, z \geqslant 0 \end{aligned}$$

The standard form of the MaxFlow problem is

$$\begin{array}{ll} \max_{f,r\in\mathbb{R}^{|E|},z\in\mathbb{R}} & z \\ \text{subject to} & \begin{pmatrix} \text{Id} & \text{Id} & 0 \\ \text{div} & 0 & (\mathbf{1}_t - \mathbf{1}_s) \end{pmatrix} \begin{pmatrix} f \\ r \\ z \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} \\ f,r,z \ge 0 \end{array}$$



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Given the decomposition $\{1, \ldots, n\} = B + N$ with |B| = m, we can define A_B as the submatrix of A that contains only those columns a^i with indices $i \in B$. Since A has maximal rank, we can select B such that $A_B \in \mathbb{R}^{m \times m}$ has maximal rank and we can compute

$$x_B = A_B^{-1}b$$



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 x_B only defines those entries of x whose indices are in B. Filling the rest of x with zeros ($x_N = 0$), we obtain a feasible x. Feasible x that are created in this way ($x = x_B + x_N$) are called **basic feasible solutions**.



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We have $0 = \sum_{i=1}^{k} \lambda_i a^i$ with at least one $\lambda_i > 0$ and thus for all $\epsilon > 0$

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$$b = \sum_{i=1}^{k} (x_i - \epsilon \lambda_i) a_i$$

Choosing $\epsilon = \min \left\{ \frac{x_i}{\lambda_i} \middle| \lambda_i > 0 \right\}$ creates a feasible solution $x' = x - \epsilon \lambda$ that uses at most k - 1 positive variables. Iterating this step leads eventually to the 1st case of linear independence.



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If $\langle c, \lambda \rangle \neq 0$, we could improve x^* for small ϵ , which contradicts the optimality of x^* .

Thus, $\langle c, \lambda \rangle = 0$, which proves the optimality of x'. x' is a feasible solution that uses at most k - 1 positive variables. Iterating this step eventually leads to the case of linear independence and thus, proves the theorem. \Box



Simplex Method





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The main challenge is to perform Step 2 as efficiently as possible.



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The main challenge is to perform Step 2 as efficiently as possible.

There are certain LPs for which Step 1 is difficult. For the problems we will consider, this step will be very easy.





Since the $(a_i)_{i\in B}$ form a base of \mathbb{R}^m , we also have

$$a_j = \sum_{i \in B} y_{ij} a_i$$
 for all $j \notin B$

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For $\epsilon = \min_k \left\{ \frac{x_k}{y_{kj}} \middle| y_{kj} > 0 \right\}$, x_{ϵ} becomes a basic feasible solution w.r.t. the basic set $B - \{j\} + \{k\}$ where k is the minimizing index that defines ϵ .



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and the cost function becomes for $y_N \neq 0$

$$\langle c, y \rangle = \langle y_N, c_N \rangle + \left\langle A_B^{-1} \left(b - A_N y_N \right), c_B \right\rangle$$

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We can improve the solution iff $c_N - A_N^{\top} A_B^{-\top} c_B$ has positive entries.





Given a basic feasible solution defined by B, we know whether it is optimal or not. If it is not optimal, we know how to change B in order to get an improved solution. In addition, we know how x_B will change if we change B.



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To this end, we will store an LP that is equivalent to the original LP. This representation is called the **Simplex Tableau**.



Given a basic feasible solution x_B and its basic set B, the simplex tableau is a $(m+1) \times (n+1)$ matrix of the following form

$$\begin{array}{c|c} 0 & c_N^\top - c_B^\top A_B^{-1} A_N & -\langle c_B, x_B \rangle \\ \hline \text{Id} & A_B^{-1} A_N & A_B^{-1} b \end{array}$$



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If the first row has a positive entry (at position j), we can improve the solution by adding j to B. Select $i \in \operatorname{argmin} \left\{ \frac{(A_B^{-1}b)_i}{(A_B^{-1}a^j)_i} \middle| (A_B^{-1}a^j)_i > 0 \right\}$ and pivot the j^{th} column of the tableau, i.e, perform Gaussian elimination until the j^{th} column is the $(i + 1)^{\text{th}}$ unit vector.



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This operation only changes the i^{th} column among the first m columns. In particular, one can show that we obtain a tableau with respect to B - i + j.



The pivoting operation can be summarized in the following form: At each step, there exists a vector $v \in \mathbb{R}^m$ such that the tableau is representable as the following product

$$\begin{pmatrix} 1 & -v^{\top} \\ 0 & A_B^{-1} \end{pmatrix} \cdot \begin{pmatrix} c^{\top} & 0 \\ A & b \end{pmatrix} = \begin{pmatrix} 1 & -v^{\top} \\ 0 & A_B^{-1} \end{pmatrix} \cdot \begin{pmatrix} c_B^{\top} & c_N^{\top} & 0 \\ A_B & A_N & b \end{pmatrix}$$
$$= \begin{pmatrix} 0 & c_N^{\top} - c_B^{\top} A_B^{-1} A_N & | & -\langle c_B, x_B \rangle \\ \text{Id} & A_B^{-1} A_N & | & A_B^{-1} b \end{pmatrix}$$



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It is easy to check that
$$v^{\top} = c_B^{\top} A_B^{-1}$$
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There are methods that can solve the problem in polynomial time, but are numerical less stable than the simplex method. In practice, the simplex is quite fast and does not visit every basic feasible solution.

Nonetheless, there is an LP for which the simplex method might visit every of its 2^n basic feasible solutions. For $0 < \epsilon < \frac{1}{2}$ this is such an example

$$\begin{array}{ll} \max_{x \in \mathbb{R}^n} & x_n \\ \text{subject to} & 0 \leqslant x_1 \leqslant 1 \\ \epsilon x_i \leqslant x_{i+1} \leqslant 1 - \epsilon x_i & \text{for all } i = 1, \dots, n-1 \end{array}$$



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- Dropping the integer constraints on the variable leads to an LP
- **This LP might have an optimizer** $x \in \mathbb{R}^n$
- If $x \in \mathbb{Z}^n$, the ILP is solved by x.



Let us assume that we have the following maximization problem

 $\max_{x \in \mathbb{Z}^n} \langle c, x \rangle$
subject to Ax = b
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with $b \in \mathbb{Z}^n$ and $A \in \mathbb{Z}^{m \times n}$.



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Let us assume that the LP that ignores the integer constraint will find a solution $x = x_B + x_N$ with

$$x_B = A_B^{-1}b \qquad \qquad x_N = 0.$$

Since $b \in \mathbb{Z}^m$, the ILP would be solved if $A_B^{-1} \in \mathbb{Z}^{m \times n}$. While this is not true in general, we can classify those matrices that give rise to ineger solutions.

- UID	Total	ly Unimodu	rix		
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- UP	Total	ly Unimodu	rix		
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Lemma 1. If $A \in \mathbb{Z}^{m \times n}$ is totally unimodular, we have $A_B^{-1} \in \mathbb{Z}^{m \times n}$ for any basic set $B \subset \{1, \ldots, n\}$.

N	Total	ly Unimodu	rix		
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Proof. Let $M \in \mathbb{R}^{m \times m}$ be an arbitrary invertible matrix. Then we denote by $m_{i,j}^{\#} = \det(M_{j,i})$ the determinant of the submatrix of M after removing the j^{th} row and the i^{th} column of M. This creates a new matrix $M^{\#}$ and for the product of these matrices we have $MM^{\#} = \det(M) \cdot \text{Id}$.

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If $A \in \mathbb{R}^{m \times n}$ is totally unimodular, we have $A \in \{-1, 0, 1\}^{m \times n}$.



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Given $A \in \{-1, 0, 1\}^{m \times n}$, there is a very useful classification by Ghouila-Houri of totally unimodular matrices:

Theorem 3. Iff for every selection of rows $a^i, i \in R$, there is a seperation $R = R^+ + R^-$ such that $\sum_{i \in R^+} a^i - \sum_{i \in R^-} a^i \in \{-1, 0, 1\}^n$, the matrix A is totally unimodular.



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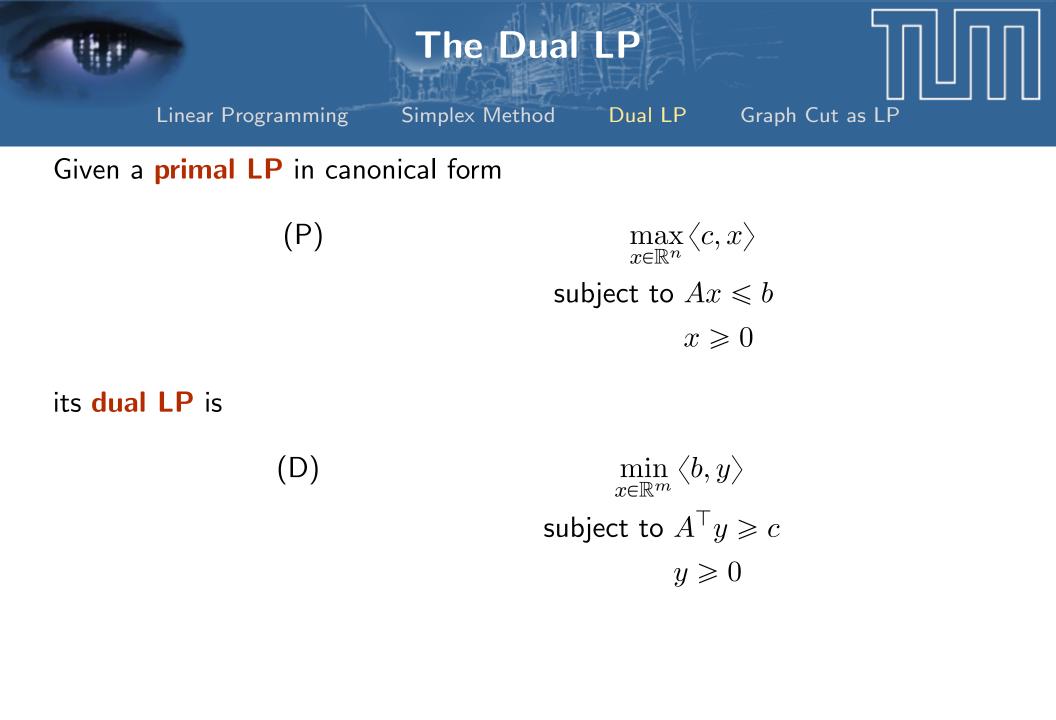
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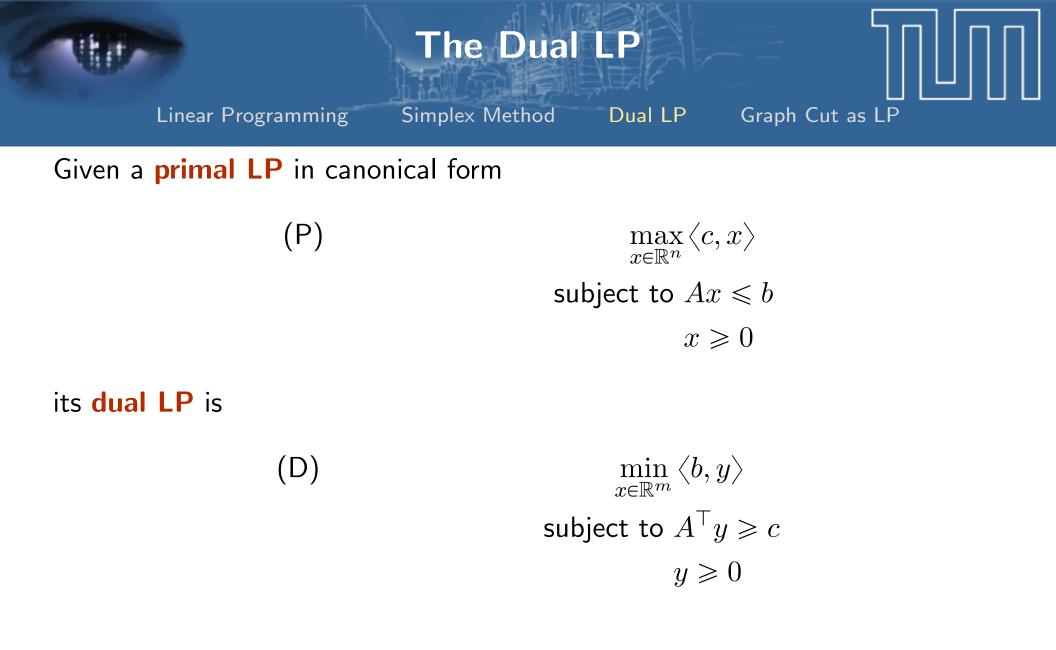
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If a matrix A is totally unimodular, A^{\top} is totally unimodular.



Dual LP





The problem is called dual due to the usage of the dual matrix A^{\top} and the dual variable $y \in \mathbb{R}^{m}$.



Proof.



Proof. Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ be feasible, i.e., $Ax \leq b$ and $c \leq A^\top y$. Then

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Let us now assume that $x^* \in \mathbb{R}^n$ is an optimizers of (P) that is obtained via the simplex method. At the last step, the simplex tableau looks like this:

$$\begin{pmatrix} 1 & -v^{\top} \\ 0 & M \end{pmatrix} \cdot \begin{pmatrix} c^{\top} & 0 & 0 \\ A & \text{Id} & b \end{pmatrix} = \begin{pmatrix} (c - A^{\top}v)^{\top} & -v^{\top} & -\langle v, b \rangle \\ * & * & * \end{pmatrix}$$



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Since x^* is the minimizer we know that $\langle v, b \rangle = \langle x^*, c \rangle$, $c \leq A^\top v$ and $v \geq 0$. Thus v is a feasible dual variable that has the same cost as x^* . Linear Programming Simplex Method Dual LP Graph Cut as LP

Graph Cut as LP



The canonical form of the MaxFlow problem is

$$\begin{aligned} \max_{f \in \mathbb{R}^{|E|}, z \in \mathbb{R}} & z \\ \text{subject to} & \begin{pmatrix} \text{Id} & 0 \\ \text{div} & (\mathbf{1}_t - \mathbf{1}_s) \\ -\text{div} & -(\mathbf{1}_t - \mathbf{1}_s) \end{pmatrix} \begin{pmatrix} f \\ z \end{pmatrix} \leqslant \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} \\ f, z \geqslant 0 \end{aligned}$$



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Note that the transposed of div is a linear mapping that maps a function $g: V \to \mathbb{R}$ to a function $\operatorname{div}^{\top} g: E \to \mathbb{R}$. In particular, we have

$$\operatorname{div}^{\top} g(i,j) = \langle \delta_{i,j}, \operatorname{div}^{\top} g \rangle = \langle \operatorname{div} \delta_{i,j}, g \rangle$$
$$= \sum_{u \in V} g(u) \left[\operatorname{div} \delta_{i,j} \right](u)$$
$$= \sum_{u \in V} g(u) \left[\sum_{(u,v) \in E} \delta_{i,j}(u,v) - \sum_{(v,u) \in E} \delta_{i,j}(v,u) \right]$$
$$= g(i) - g(j)$$



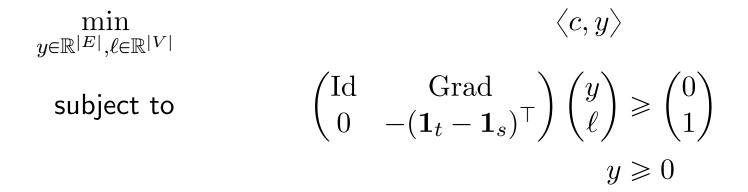
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In other words, we have $\operatorname{div}^{\top} = -\operatorname{Grad}$.



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$$\min_{\substack{y \in \mathbb{R}^{|E|}, \ell \in \mathbb{R}^{|V|}}} & \langle c, y \rangle \\ \text{subject to} & \begin{pmatrix} \text{Id} & \text{Grad} \\ 0 & -(\mathbf{1}_t - \mathbf{1}_s)^\top \end{pmatrix} \begin{pmatrix} y \\ \ell \end{pmatrix} \ge \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & y \ge 0 \end{pmatrix}$$

Since the constraint matrix is totally unimodular, we obtain

$$\begin{array}{ll} \min_{y \in \mathbb{Z}^{|E|}, \ell \in \mathbb{Z}^{|V|}} & \langle c, y \rangle \\ \text{subject to} & y \geqslant - \operatorname{Grad}(\ell) \\ & \ell(s) \geqslant \ell(t) + 1 \\ & y \geqslant 0 \end{array}$$



We make the following observations:

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- For $\ell(s) > 1$, the cost w.r.t. $\left(\frac{\ell}{\ell(s)}, \frac{y}{\ell(s)}\right)$ is lower. Hence $\ell(s) := 1$.
- If there exists a node $i \in V$ with $\ell(i) \notin \{0, 1\}$, we can decrease the cost by clipping ℓ at 0 resp. 1.

Hence, the dual of the MaxFlow is the MinCut:

$\min_{\ell\in\mathbb{B}^{ V }}$	$\langle c, \max(0, -$	$\operatorname{Grad}\ell)\rangle$
subject to	$\ell(s) = 1$	$\ell(t) = 0$





We saw that quadratic submodular pseudo-Boolean functions can be cast as a graph cut problem with positive edge weights. MaxFlow, the dual problem of MinCut, exploits this non-negativity.



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Next lecture we will reformulate the general quadratic pseudo-Boolean optimization problem in order to minimize some non-submodular energies.



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