Combinatorial Optimization in Computer Vision (IN2245)

Frank R. Schmidt
Csaba Domokos

Winter Semester 2015/2016

8. Roof Duality

QPBO

Quadratic Pseudo-Boolean Energies

We saw that every pseudo-Boolean energy $E\colon \mathbb{B}^n \to \mathbb{R}$ can be represented as a multilinear function in the variables $x_1, \dots, x_n$.

In the following we assume the case of **quadratic pseudo-Boolean** energies

$$E(x) = C_0 + \sum_{i=1}^{n} C_i x_i + \sum_{i,j=1}^{n} C_{ij} x_i x_j$$

$C_{ij} < 0$ refer to submodular terms and $C_{ij} > 0$ to supermodular terms.

Defining the sets

$$N := \left\{ (i,j) \in \{1, \dots, n\}^2 \middle| C_{ij} < 0 \right\}$$
$$P := \left\{ (i,j) \in \{1, \dots, n\}^2 \middle| C_{ij} > 0 \right\},$$

we know that E is submodular (supermodular) iff $|P| = 0$ ($|N| = 0$).

Quadratic Pseudo-Boolean Optimization

For submodular functions

$$E(x) = C_0 + \sum_{i=1}^{n} C_i x_i + \sum_{(i,j)\in N} C_{ij} x_i x_j,$$

we saw that the problem can be cast as a MaxFlow problem that can be solved efficiently. In particular, the MaxFlow-MinCut theorem is a special case of the duality theorem for LPs.

What makes the MaxFlow-MinCut duality so interesting is that while MaxFlow is an LP, the MinCut is an ILP. The strong duality between these two problems is only possible because the constraint matrix is totally unimodular.

The idea of the **quadratic pseudo-Boolean optimization (QPBO)** is to reformulate the minimization problem as an ILP and to find an approximative solution. Since QPBO is NP hard, we cannot expect to find the minimal energy. Instead we will compute a lower bound of the minimum energy.

Linearization

Linearization

The central idea of transforming a quadratic energy into a linear energy is to introduce new variables $y_{ij} \in [0,1]$ and adding extra *linear constraints* in order to assure that for $x_i, x_j \in \mathbb{B}$, we have $y_{ij} = x_i \cdot x_j$.

If either $x_i = 0$ or $x_j = 0$, we have to enforce $y_{ij} = 0$. This can be done by

$$y_{ij} \leqslant x_i \qquad\qquad y_{ij} \leqslant x_j$$

Note that for $x_i = x_j = 1$ these constraints have no effect on y_{ij}.

If $x_i = x_j = 1$, we have to enforce $y_{ij} = 1$. This can be done by

$$x_i + x_j - 1 \leqslant y_{ij}$$

Note that for $x_i = 0$ or $x_j = 0$ this constraint has no effect on y_{ij}.

Discrete Rhys Form

Summarizing these observations, we obtain the following ILP for QPBO:

$$\text{(RF)} \quad \min_{x\in\mathbb{R}^n, y\in\mathbb{R}^{n\times n}} \quad \sum_{i=1}^{n} C_i x_i + \sum_{(i,j)\in P+N} C_{ij} y_{ij}$$

$$\begin{aligned}
\text{subject to:} \quad & y_{ij} - x_i - x_j \geqslant -1 && \text{for all } (i,j) \in P \\
& x_i - y_{ij} \geqslant 0 && \text{for all } (i,j) \in N \\
& x_j - y_{ij} \geqslant 0 && \text{for all } (i,j) \in N \\
& -x \geqslant -1 && \\
& x, y \geqslant 0 && \\
& x \in \mathbb{Z}^n &&
\end{aligned}$$

Note that due to our energy it is not necessary to enforce all three constraints to all y_{ij}. The constraint matrix becomes totally unimodular if $P = \emptyset$.

Continuous Rhys Form

Instead of solving the **Discrete Rhys Form**, we can just solve the **Continuous Rhys Form** by dropping the integer constaint $x \in \mathbb{Z}^n$.

Technically, we are minimizing the same energy. The only thing we change is the set of feasible points. Hence we have for the LP's minimizer $(\hat{x}, \hat{y}) \in \mathbb{R}^n$

$$\begin{aligned} & C_0 + \sum_{i=1}^{n} C_i \hat{x}_i + \sum_{(i,j) \in P+N} C_{ij} \hat{x}_i \hat{x}_j \\ \leqslant & C_0 + \sum_{i=1}^{n} C_i x_i + \sum_{(i,j) \in P+N} C_{ij} x_i x_j \qquad \text{for all } x \in \mathbb{B}^n. \end{aligned}$$

We denote the minimum of the continuous Rhys form as $L(E)$ and we know

$$L(E) \leqslant \min_{x \in \mathbb{B}^n} E(x)$$

Minorization

Modular Minorants

Given a quadratic pseudo-Boolean function $E \colon \mathbb{B}^n \to \mathbb{R}$, we call a modular function $f \colon \mathbb{B}^n \to \mathbb{R}$ a **modular minorant** if

$$f(x) = f_0 + \sum_{i=1}^{n} f_i x_i \leqslant E(x) \qquad \text{for all } x \in \mathbb{B}^n$$

We denote the set of modular functions as $\mathcal{F}_1$.

A family $\mathcal{R} \subset \mathcal{F}_1$ is called **complete** iff

$$E(x) = \max_{f \in \mathcal{R}} f(x) \qquad \text{for all } x \in \mathbb{B}^n$$

Given a complete family $\mathcal{R} \subset \mathcal{F}_1$, we can formulate the QPBO as

$$\min_{x \in \mathbb{B}^n} E(x) = \min_{x \in \mathbb{B}^n} \max_{f \in \mathcal{R}} f(x)$$

Complete $\mathcal{F}_1$-Family for Monomials

Since the sum of modular functions are modular, it is enough to provide a complete Family $\mathcal{R} \subset \mathcal{F}_1$ for the quadratic energies xy and $-xy$.

In order to satisfy $\alpha + \beta x + \gamma y \leqslant \pm xy$, the following has to be satisfied

$$\alpha, \alpha + \beta, \alpha + \gamma \leqslant 0 \qquad \alpha + \beta + \gamma \leqslant \pm 1$$

Minimizing the sum of the gaps (L^1 distance) leads to

$$\mathcal{R}(xy) = \{\lambda(x + y - 1) | \lambda \in [0,1]\}$$

and

$$\mathcal{R}(-xy) = \{-[\lambda x + (1-\lambda) y] | \lambda \in [0,1]\}.$$

Roof Duality

Combining the information of monomials, we obtain

$$\mathcal{R}(E) = \left\{ \begin{array}{c} C_0 + \sum_{i=1}^{n} C_i x_i + \\ \sum_{(i,j) \in P} C_{ij} \lambda_{ij} [x_i + x_j - 1] + \\ \sum_{(i,j) \in N} C_{ij} [\lambda_{ij} x_i + (1 - \lambda_{ij}) x_j] \end{array} \middle| \lambda_{ij} \in [0,1] \right\}$$

as a complete family of modular minorants.

We can now define another lower bound for $\min_{x \in \mathbb{B}^n} E(x)$

$$M(E) = \max_{f \in \mathcal{R}(E)} \min_{x \in \mathbb{B}^2} f(x) \leqslant \min_{x \in \mathbb{B}^2} \max_{f \in \mathcal{R}(E)} f(x) = \min_{x \in \mathbb{B}^n} E(x).$$

This lower value is called the **roof dual of E**.

The **roof duality** was originally introduced for maximization problems. The term "roof" reflects this, since it was defined as modular majorization.

Roof Duality

Every $f \in \mathcal{R}$ is modular and therefore we can compute its minimum. Let

$$\begin{aligned} f(x) =& C_0 + \sum_{i=1}^{n} C_i x_i + \\ & \sum_{(i,j) \in P} C_{ij} \lambda_{ij} [x_i + x_j - 1] + \sum_{(i,j) \in N} C_{ij} [\lambda_{ij} x_i + (1 - \lambda_{ij}) x_j] \\ =& \underbrace{\left[C_0 - \textstyle\sum_{(i,j) \in P} \lambda_{ij} C_{ij} \right]}_{\nu_0(f)} + \\ & \sum_{i=1}^{n} \underbrace{\left[C_i + \textstyle\sum_{j=1}^{n} \lambda_{ij} C_{ij} + \sum_{j,(j,i) \in P} \lambda_{ji} C_{ji} + \sum_{j,(j,i) \in N} (1 - \lambda_{ji}) C_{ji} \right]}_{\nu_i(f)} x_i \end{aligned}$$

$$M(E) = \max_{f \in \mathcal{R}(E)} \nu_0(f) + \sum_{i=1}^{n} \nu_i(f)^- \qquad \text{where } v^- := \min(v, 0)$$

Roof Duality as LP *

We can now formulate the roof duality as an LP:

$$\begin{aligned} \text{(RD)} \quad \max_{\Lambda, M \in \mathbb{R}^{n \times n}, \epsilon \in \mathbb{R}^n} \quad & \sum_{(i,j) \in P} -\Lambda_{ij} + \sum_{i=1}^{n} -\epsilon_i \\ \text{subject to} \quad & \Lambda_{ij} \leqslant C_{ij} \quad \text{for } (i,j) \in P \\ & -(\Lambda_{ij} + M_{ij}) \leqslant C_{ij} \quad \text{for } (i,j) \in N \\ & -\sum_{j,(i,j) \in P} \Lambda_{ij} - \sum_{j,(j,i) \in P} \Lambda_{ji} + \\ & \sum_{j,(i,j) \in N} \Lambda_{ij} + \sum_{j,(j,i) \in N} M_{ji} - C_i \leqslant \epsilon_i \quad \text{for } i = 1, \dots, n \\ & \Lambda, M, \epsilon \geqslant 0 \end{aligned}$$

Since this LP (RD) is dual to the Rhys form LP (RF), we have $M(E) = L(E)$.

Complementation

Posiforms

Hammer showed that these relaxations of a quadratic pseudo-Boolean energy are equivalent. Besides these two interpretations Hammer proposed a third interpretation that gives rise to an easy algorithm to compute the roof duality.

A pseudo-Boolean energy E can be written as a quadratic posiform $\Phi \in \mathcal{P}_2$. The constant $C_0(\Phi)$ of such as posiform provides us with a lower bound for E. Since there is not a unique posiform representation for E, we can look for a posiform with maximal $C_0(\Phi)$. This provides us with a third lower bound

$$C(E) = \max_{\Phi \in \mathcal{P}_2} C_0(\Phi) \leqslant \min_{x \in \mathbb{B}^n} E(x).$$

The MaxFlow method can be seen as rewriting a submodular pseudo-Boolean function in different posiforms until we obtain a *proof* for

$$\exists \Phi \in \mathcal{P}_2 : E - \Phi = \min_{x \in \mathbb{B}^n} E(x).$$

This equality is not true for general quadratic pseudo-Boolean functions.

Roof Duality and Posiforms *

Lemma 1. *For a quadratic $E\colon \mathbb{B}^n \to \mathbb{R}$ we have $M(E) \leqslant C(E)$.*

Proof. Let $p = \nu_0 + \sum_{i=1}^n \nu_i x_i \in \mathcal{R}(E)$ be a "roof" of E.
Then we know that $E - p$ is a positive linear combination of

$$\begin{aligned}
-x_i x_j + [(1-\lambda_{ij})x_i + \lambda_{ij}x_j] &= \lambda_{ij} x_i \overline{x}_j + (1-\lambda_{ij})\overline{x}_i x_j \\
x_i x_j - \lambda_{ij}[x_i + x_j - 1] &= \lambda_{ij}\overline{x}_i\overline{x}_j + (1-\lambda_{ij}) x_i x_j
\end{aligned}$$

Hence $\Phi := E - p \in \mathcal{P}_2$ is a posiform. Rewriting $p(x) = L(x, \overline{x}) + c$ with

$$c = \nu_0 + \sum_{i=1}^n \nu_i^- \qquad L(x,\overline{x}) = \sum_{i=1}^n \nu_i^+ x_i - \sum_{i=1}^n \nu_i^- \overline{x}_i$$

we have $E = c + (L + \Phi)$ with $L + \Phi \in \mathcal{P}_2$.
Since $c = \min_{x \in \mathbb{B}^n} p(x)$, we have $M(E) \leqslant C(E)$.

Roof Duality and Posiforms *

Lemma 2. *For a quadratic $E\colon \mathbb{B}^n \to \mathbb{R}$ we have $C(E) \leqslant M(E)$.*

Proof. Given the representation $E = C(E) + \Phi$ for $\Phi \in \mathcal{P}_2$, we can write $\Phi = \Phi_1 + \Phi_2$ where Φ_1 contains the linear and Φ_2 the quadratic part of Φ:

$$\Phi_2(x) = \sum_{i,j=1}^n \alpha_{ij} x_i x_j + \beta_{ij}\overline{x}_i\overline{x}_j + \gamma_{ij}\overline{x}_i x_j + \delta_{ij} x_i \overline{x}_j \qquad (\alpha,\beta,\gamma,\delta \geqslant 0)$$

Observing that $xy + x\overline{y} = x$, we can choose the seperation of Φ_1 and Φ_2 in a way that $\alpha_{ij} + \beta_{ij} > 0$ and $\gamma_{ij} + \delta_{ij} > 0$ is not simultaneously true. Defining

$$\begin{aligned}
P :=& \{(i,j) \in \{1,\dots,n\}^2 | \alpha_{ij} + \beta_{ij} > 0\} \\
N :=& \{(i,j) \in \{1,\dots,n\}^2 | \gamma_{ij} + \delta_{ij} > 0\}
\end{aligned}$$

we can rewrite the quadratic terms of $p := E - \Phi_2 = C(E) + \Phi_1$ as

Roof Duality and Posiforms *

Proof (Cont.).

$$0 = \sum_{(i,j) \in P}^{n} (C_{ij} - \alpha_{ij} - \beta_{ij})\, x_i x_j + \sum_{(i,j) \in N}^{n} (C_{ij} + \gamma_{ij} + \delta_{ij})\, x_i x_j$$

Hence $C_{ij} = \alpha_{ij} + \beta_{ij}$ for $(i,j) \in P$ and $-C_{ij} = \gamma_{ij} + \delta_{ij}$ for $(i,j) \in N$. Defining $\lambda_{ij} = \frac{\beta_{ij}}{C_{ij}}$ for $(i,j) \in P$ and $\lambda_{ij} = \frac{\delta_{ij}}{-C_{ij}}$ for $(i,j) \in N$ we obtain

$$\begin{aligned}
p =& C_0 + \sum_{i=1}^n C_i x_i + \\
& \sum_{(i,j) \in P} C_{ij}\lambda_{ij}[1 - x_i - x_j] + \sum_{(i,j) \in N} C_{ij}[\lambda_{ij} x_i + (1-\lambda) x_j]
\end{aligned}$$

Thus $p = C(E) + \Phi_1 \in \mathcal{R}(E)$ and therefore $M(E) \geqslant C(E)$.

Equivalent Roof Duality Formulations

Theorem 1 (Hammer et al.). *For a pseudo-Boolean function E, the roof duality is $M(E) = L(E) = C(E)$.*

This equivalence is surprising, because each lower bound is interpreting the energy E very differently

- The **minorization** interprets E as a Pseudo-Boolean functions and approximates it from below by modular functions.
- The **linearization** interprets the minimization of E as an ILP and uses the continuous relaxation by ignoring the integer constraints.
- The **complementation** interprets E as an algebraic expression and performs algebraic reformulations.

$L(E) = M(E)$ proves that the roof duality can be solved in polynomial time by optimizing an LP. $C(E) = M(E)$ will help us to reformulate the computation of the roof duality as a MaxFlow problem.

Network Flow

Roof Duality as Network Flow

Given a quadratic pseudo-Boolean energy in posiform

$$\Phi(x) = C_0 + \sum_{i \in \mathcal{L}} C_i x_i + \sum_{i,j \in \mathcal{L}} C_{ij} x_i x_j$$

where $\mathcal{L} = \{1, \dots, n, \overline{1}, \dots, \overline{n}\}$ is the set of literals and $x_{\overline{i}} := \overline{x}_i = 1 - x_i$.

Now we define the network $G_\Phi = (V, E, c, 0, \overline{0})$ with $V = \mathcal{L} + \{0, \overline{0}\}$. Using the conventions that $x_0 = 1$ and $x_{\overline{0}} = 0$ we can rewrite Φ as

$$\Phi(x) = C_0 + \sum_{i,j \in V} C_{ij} x_i x_j \qquad C_{0i} := C_i$$

The edges $(i,j) \in E$ and the capacities $c\colon E \to \mathbb{R}_0^+$ are defined as

$$C_{ij} > 0 :\Leftrightarrow (i,\overline{j}), (j,\overline{i}) \in E \qquad c(i,\overline{j}) = c(j,\overline{i}) = \frac{1}{2} C_{ij}$$

Computation of Roof Duality

Computing the maximal flow in the network G_Φ changes the capacities of the residual graph. Let $i_0, \dots, i_k$ be a path from $i_0 = 0$ to $i_k = \overline{0}$. Then:

$$\Phi' = \Phi - \epsilon\left[x_{i_1} + \sum_{j=1}^{k-1} x_{i_j}\overline{x}_{i_{j+1}} + \overline{x}_{i_k}\right] + \epsilon \sum_{j=1}^{k-1} \overline{x}_{i_j} x_{i_{j+1}} + \epsilon$$

This proves that we can compute a lower bound of the roof duality.

Since there is a one-to-one relationship between posiforms Φ and networks G_Φ, we obtain equality. In other words, the max flow provides us with the value of the roof duality.

Strong Persistency

Theorem 2. *Let $E\colon \mathbb{B}^n \to \mathbb{R}$ be a pseudo-Boolean function represented as a posiform Φ such that 0 and $\overline{0}$ are disconnected in G_Φ and let $S \subset \mathcal{L}$ be the set of literals that are path-connected with the source 0.*
Given an arbitrary $x \in \mathbb{B}^n$, we can create x_S that replaces each literal in S with the value 1. Then we have $E(x_S) \leqslant E(x)$.

Before we prove this theorem, we have to see whether x_S is well defined. To this end we have to prove that for each $u \in S$ we have $\overline{u} \notin S$. Assume that $u, \overline{u} \in S$. In other words there are paths from 0 to u and from 0 to $\overline{u}$ in G_Φ.

$$\sum_{j=0}^{k} x_{i_j}\overline{x}_{i_{j+1}} = \sum_{j=0}^{k} x_{\overline{i}_{k-j+1}}\overline{x}_{\overline{i}_{k-j}} \qquad i_0 = 0, i_{k+1} = \overline{u}$$

This shows that the path from 0 to $\overline{u}$ implies a path from u to $\overline{0}$.
Together with the path from 0 to u we constructed a contradiction.

Strong Persistency

Proof (Strong Persistency). Since x_S is well defined, we also know that the inversed literals of S are in the connected component T with respect to $\overline{0}$.
Every $x \in \mathbb{B}^n$ defines a cut (A, B) of the graph G_Φ with

$$E(x) = C(E) + \mathrm{Cut}(A, B).$$

By replacing A with $A' = (A \cup S)\backslash T$ and B with $B' = (B \cup T)\backslash S$, we obtain a lower cut value. Moreover (A', B') is the cut associated with x_S and the theorem is proven.

The persistency theorem shows that we can find the correct labeling for some of the involved variables. By replacing these variables with their true values, we can reformulate the energy with respect to the remaining $n - |S|$ variables. This procedure can ber iterated until $|S| = 0$.

Kolmogorov's Method

Kolmogorv proposed a certain set of heuristics in order to obtain as much information as possible from the roof duality computation:

- First compute the maximal flow in the subgraph G_0 that only contains the submodular pairwise terms.
- Create the roof duality graph with respect to the residual graph of G_0 and add the edges with respect to the supermodular terms.
- After computing the maximal flow label all nodes u such that there is no path from u to $\overline{u}$ in the residual graph. This is done by analyzing the connected components of $V - S - T$.
- The connected components form an acyclic digraph and we obtain a topological ordering $\pi : V - S - T \to \mathbb{Z}$ which helps us to extend S and T:

$$\begin{aligned} S' &= S + \{u \mid \pi(u) > \pi(\overline{u})\} \\ T' &= T + \{u \mid \pi(u) < \pi(\overline{u})\} \end{aligned}$$

Literature *

Roof Duality

- Rhys, *"A selection problem of shared fixed costs and networks."*, 1970, Management Science (17), 200–207.
- Hammer, Hansen, Simeone, *Roof Duality, Complementation and Persistency in Quadratic 0-1 Optimization*, 1984, Math. Programming 28 (2), 121–155.
- Boros and Hammer, *Pseudo-Boolean Optimization*, 2002, Discrete Applied Mathematics (123), 155–225.

Computer Vision

- Kolmogorov, Rother, *Minimizing non-submodular Functions with Graph Cuts*, 2007, IEEE TPAMI 29(7), 1274–1279