

Combinatorial Optimization in Computer Vision (IN2245)

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9. Belief Propagation

Inference revisited

Inference means the procedure to estimate the probability distribution, encoded by a graphical model, for a given data.

Assume we are given a factor graph and the observation x .

- **Maximum A Posteriori (MAP) inference:** find the state $y^* \in \mathcal{Y}$ of maximum probability,

$$y^* \in \operatorname{argmax}_{y \in \mathcal{Y}} p(Y = y | x) = \operatorname{argmin}_{y \in \mathcal{Y}} E(y; x).$$

- **Probabilistic inference:** find the value of the *log partition function* and the *marginal distributions* for each factor,

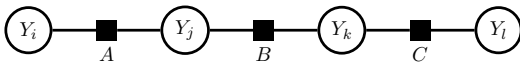
$$\log Z(x) = \log \sum_{y \in \mathcal{Y}} \exp(-E(y; x)),$$

$$\mu_F(y_F) = p(Y_F = y_F | x) \quad \forall F \in \mathcal{F}, \forall y_F \in \mathcal{Y}_F.$$

Sum-product algorithm

Inference on chains

Assume that we are given the following factor graph and a corresponding energy function $E(y)$, where $\mathcal{Y} = \mathcal{Y}_i \times \mathcal{Y}_j \times \mathcal{Y}_k \times \mathcal{Y}_l$.



We want to compute $p(y)$ for any $y \in \mathcal{Y}$ by making use of the factorization

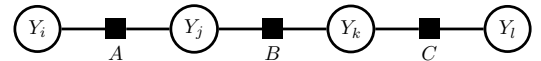
$$p(y) = \frac{1}{Z} \exp(-E(y)).$$

Problem: we also need to calculate the *partition function*

$$Z = \sum_{y \in \mathcal{Y}} \exp(-E(y)) = \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \sum_{y_k \in \mathcal{Y}_k} \sum_{y_l \in \mathcal{Y}_l} \exp(-E(y_i, y_j, y_k, y_l)),$$

which looks expensive (the sum has $|\mathcal{Y}_i| \cdot |\mathcal{Y}_j| \cdot |\mathcal{Y}_k| \cdot |\mathcal{Y}_l|$ terms).

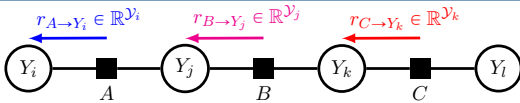
Inference on chains (cont.)



We can expand the *partition function* as

$$\begin{aligned} Z &= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \sum_{y_k \in \mathcal{Y}_k} \sum_{y_l \in \mathcal{Y}_l} \exp(-E(y_i, y_j, y_k, y_l)) \\ &= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \sum_{y_k \in \mathcal{Y}_k} \sum_{y_l \in \mathcal{Y}_l} \exp(-E_A(y_i, y_j) + E_B(y_j, y_k) + E_C(y_k, y_l)) \\ &= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \sum_{y_k \in \mathcal{Y}_k} \sum_{y_l \in \mathcal{Y}_l} \exp(-E_A(y_i, y_j)) \exp(-E_B(y_j, y_k)) \exp(-E_C(y_k, y_l)) \\ &= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) \sum_{y_k \in \mathcal{Y}_k} \exp(-E_B(y_j, y_k)) \sum_{y_l \in \mathcal{Y}_l} \exp(-E_C(y_k, y_l)). \end{aligned}$$

Inference on chains (cont.)

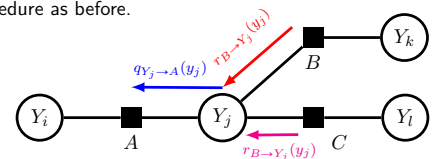


Note that we can successively *eliminate* variables, that is

$$\begin{aligned} Z &= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) \sum_{y_k \in \mathcal{Y}_k} \exp(-E_B(y_j, y_k)) \sum_{y_l \in \mathcal{Y}_l} \exp(-E_C(y_k, y_l)) \\ &= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) \sum_{y_k \in \mathcal{Y}_k} \exp(-E_B(y_j, y_k)) r_{C \rightarrow Y_k}(y_k) \\ &= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) r_{B \rightarrow Y_j}(y_j) = \sum_{y_i \in \mathcal{Y}_i} r_{A \rightarrow Y_i}(y_i). \end{aligned}$$

Inference on trees

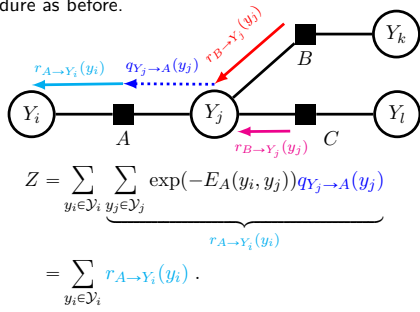
Now we are assuming a tree-structured factor graph and applying the same elimination procedure as before.



$$\begin{aligned} Z &= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) \sum_{y_k \in \mathcal{Y}_k} \exp(-E_B(y_j, y_k)) \sum_{y_l \in \mathcal{Y}_l} \exp(-E_C(y_j, y_l)) \\ &= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) r_{B \rightarrow Y_j}(y_j) r_{C \rightarrow Y_j}(y_j) \\ &= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) q_{Y_j \rightarrow A}(y_j) \end{aligned}$$

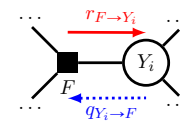
Inference on trees (cont.)

Now we are assuming a tree-structured factor graph and applying the same elimination procedure as before.



Messages

Message: pair of vectors at each factor graph edge $(i, F) \in \mathcal{E}$.

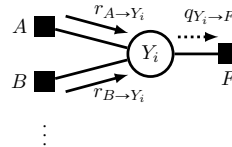


- Variable-to-factor message** $q_{Y_i \rightarrow F} \in \mathbb{R}^{\mathcal{Y}_i}$ is given by

$$q_{Y_i \rightarrow F}(y_i) = \prod_{F' \in M(i) \setminus \{F\}} r_{F' \rightarrow Y_i}(y_i),$$

where $M(i) = \{F \in \mathcal{F} : (i, F) \in \mathcal{E}\}$ denotes the set of factors adjacent to Y_i .

- Factor-to-variable message:** $r_{F \rightarrow Y_i} \in \mathbb{R}^{\mathcal{Y}_i}$.

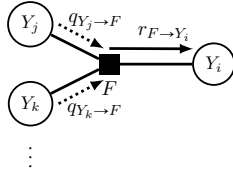


Messages (cont.)

- Factor-to-variable message** $r_{F \rightarrow Y_i} \in \mathbb{R}^{\mathcal{Y}_i}$ is given by

$$r_{F \rightarrow Y_i}(y_i) = \sum_{\substack{y'_F \in \mathcal{Y}_F \\ y'_i = y_i}} \left(\exp(-E_F(y'_F)) \prod_{l \in N(F) \setminus \{i\}} q_{Y_l \rightarrow F}(y'_l) \right),$$

where $N(F) = \{i \in V : (i, F) \in \mathcal{E}\}$ denotes the set of variables adjacent to F .



Message scheduling

One can remark that the message updates depend on each other.

$$r_{F \rightarrow Y_i}(y_i) = \sum_{\substack{y'_F \in \mathcal{Y}_F \\ y'_i = y_i}} \left(\exp(-E_F(y'_F)) \prod_{l \in N(F) \setminus \{i\}} q_{Y_l \rightarrow F}(y'_l) \right) \quad (1)$$

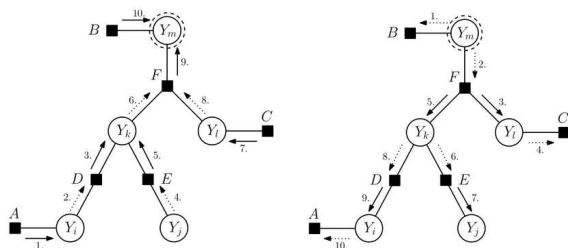
$$q_{Y_i \rightarrow F}(y_i) = \prod_{F' \in M(i) \setminus \{F\}} r_{F' \rightarrow Y_i}(y_i) \quad (2)$$

The only messages that do not depend on previous computation are the following.

- The factor-to-variable messages in which no other variable is adjacent to the factor; then the product in (1) will be empty.
- The variable-to-factor messages in which no other factor is adjacent to the variable; then the product in (2) is empty and the message will be zero.

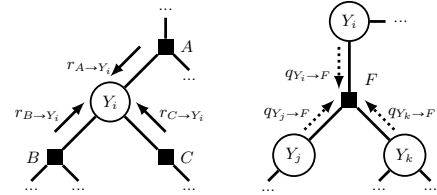
Message scheduling on trees

For tree-structured factor graphs there always exist at least one such message that can be computed initially, hence all the dependencies can be resolved.



- Select one variable node as root of the tree (e.g., Y_m)
- Compute leaf-to-root messages (e.g., by applying depth-first-search)
- Compute root-to-leaf messages (reverse order as before)

Inference result: Z and the marginals



Partition function is evaluated at the root node

$$Z = \sum_{y_i \in \mathcal{Y}_i} \prod_{F \in M(i)} r_{F \rightarrow Y_i}(y_i).$$

The marginal distribution for each factor can be computed as

$$\mu_F(y_F) = p(y_F) = \frac{1}{Z} \exp(-E_F(y_F)) \prod_{i \in N(F)} q_{Y_i \rightarrow F}(y_i).$$

Optimality and complexity

The ordering \leq over the vertex set V of a directed acyclic graph is called **topological ordering**, if for each $s \in V$, we have $t \leq s$ for all $t \in \pi(s)$, where $\pi(s)$ denotes the set of all parents of node s .

Assume a tree-structured factor graph. If the messages are computed in a topological order for the sum-product algorithm, then it converges after $2|V|$ iterations and provides the exact marginals.

If $|\mathcal{Y}_i| \leq m$ for all $i \in V$, then the complexity of the algorithm $\mathcal{O}(|V| \cdot m^2)$.

Reminder. Assuming $f, g : \mathbb{R} \rightarrow \mathbb{R}$, the notation $f(x) = \mathcal{O}(g(x))$ means that there exists $C > 0$ and $x_0 \in \mathbb{R}$ such that $|f(x)| \leq C|g(x)|$ for all $x > x_0$.

Max-sum algorithm

$$y^* \in \operatorname{argmax}_{y \in \mathcal{Y}} p(y) = \operatorname{argmax}_{y \in \mathcal{Y}} \frac{1}{Z} \tilde{p}(y) = \operatorname{argmax}_{y \in \mathcal{Y}} \tilde{p}(y).$$

Similar to the *sum-product algorithm* one can obtain the so-called **max-sum algorithm** to solve the above maximization.

By applying the \ln function, we have

$$\begin{aligned} \ln \max_{y \in \mathcal{Y}} \tilde{p}(y) &= \max_{y \in \mathcal{Y}} \ln \tilde{p}(y) \\ &= \max_{y \in \mathcal{Y}} \ln \prod_{F \in \mathcal{F}} \exp(-E_F(y_F)) \\ &= \max_{y \in \mathcal{Y}} \sum_{F \in \mathcal{F}} -E_F(y_F). \end{aligned}$$

The messages become as follows

$$\begin{aligned} q_{Y_i \rightarrow F}(y_i) &= \sum_{F' \in M(i) \setminus \{F\}} r_{F' \rightarrow Y_i}(y_i) \\ r_{F \rightarrow Y_i}(y_i) &= \max_{\substack{y'_F \in \mathcal{Y}_F, \\ y'_i = y_i}} \left(-E_F(y'_F) + \sum_{l \in N(F) \setminus \{i\}} q_{Y_l \rightarrow F}(y'_l) \right). \end{aligned}$$

The max-sum algorithm provides exact MAP inference for tree-structured factor graphs. In general, for graphs with cycles there is no guarantee for convergence.

Choosing an optimal state

First we define the **singleton max-marginal** as

$$v_i(y_i) = \max_{y' \in \mathcal{Y}, y'_i = y_i} p(y').$$

The following back-tracking algorithm is applied for choosing an optimal y^* .

1. Initialize the procedure at the root node (Y_i) by choosing any $y_i^* \in \operatorname{argmax}_{y_i \in \mathcal{Y}_i} v_i(y_i)$ and set $\mathcal{I} = \{i\}$.
2. In a topological order, for each $j \in V \setminus \{i\}$ choose a configuration y_j^* at the node Y_j such that

$$y_j^* \in \operatorname{argmax}_{y_j \in \mathcal{Y}_j} \max_{\substack{y'_j \in \mathcal{Y}_j, \\ y'_j = y_j, \forall i \in \mathcal{I} \\ y'_i = y_i^*}} p(y'),$$

and set $\mathcal{I} = \mathcal{I} \cup \{j\}$.

Sum-product and Max-sum comparison

■ Sum-product algorithm

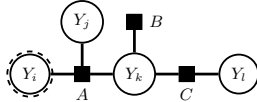
$$\begin{aligned} q_{Y_i \rightarrow F}(y_i) &= \prod_{F' \in M(i) \setminus \{F\}} r_{F' \rightarrow Y_i}(y_i) \\ r_{F \rightarrow Y_i}(y_i) &= \sum_{\substack{y'_F \in \mathcal{Y}_F, \\ y'_i = y_i}} \left(\exp(-E_F(y'_F)) \prod_{l \in N(F) \setminus \{i\}} q_{Y_l \rightarrow F}(y'_l) \right) \end{aligned}$$

■ Max-sum algorithm

$$\begin{aligned} q_{Y_i \rightarrow F}(y_i) &= \sum_{F' \in M(i) \setminus \{F\}} r_{F' \rightarrow Y_i}(y_i) \\ r_{F \rightarrow Y_i}(y_i) &= \max_{\substack{y'_F \in \mathcal{Y}_F, \\ y'_i = y_i}} \left(-E_F(y'_F) + \sum_{l \in N(F) \setminus \{i\}} q_{Y_l \rightarrow F}(y'_l) \right) \end{aligned}$$

Example *

Let us consider the following factor graph with binary variables:



$E_A(0, y_j, y_k)$	$E_A(1, y_j, y_k)$	$E_B(y_k)$	$E_C(y_k, y_l)$
$\begin{matrix} & y_k & \\ y_j & 0 & 1 \\ & 1 & 0 \end{matrix}$	$\begin{matrix} & y_k & \\ y_j & 0 & 1 \\ & 1 & 0 \end{matrix}$	$\begin{matrix} & y_k & \\ & 0 & 1 \\ 1 & 0.5 & \end{matrix}$	$\begin{matrix} & y_l & \\ y_k & 0 & 0.5 \\ & 1 & 0.5 & 0 \end{matrix}$

Let us chose the node Y_i as root. We calculate the messages for the max-sum algorithm from leaf-to-root direction in a topological order as follows.

1. $q_{Y_i \rightarrow C}(0) = q_{Y_i \rightarrow C}(1) = 0$
2. $r_{C \rightarrow Y_k}(0) = \max_{y_l \in \{0,1\}} \{-E_c(0, y_l) + q_{Y_i \rightarrow C}(0)\} = \max_{y_l \in \{0,1\}} -E_c(0, y_l) = 0$
 $r_{C \rightarrow Y_k}(1) = \max_{y_l \in \{0,1\}} \{-E_c(1, y_l) + q_{Y_i \rightarrow C}(1)\} = \max_{y_l \in \{0,1\}} -E_c(1, y_l) = 0$
3. $r_{B \rightarrow Y_k}(0) = -1$
 $r_{B \rightarrow Y_k}(1) = -0.5$
4. $q_{Y_k \rightarrow A}(0) = r_{B \rightarrow Y_k}(0) + r_{C \rightarrow Y_k}(0) = -1 + 0 = -1$
 $q_{Y_k \rightarrow A}(1) = r_{B \rightarrow Y_k}(1) + r_{C \rightarrow Y_k}(1) = -0.5 + 0 = -0.5$

Example (cont.) *

5. $q_{Y_j \rightarrow A}(0) = q_{Y_j \rightarrow A}(1) = 0$
6. $r_{A \rightarrow Y_i}(0) = \max_{y_j, y_k \in \{0,1\}} \{-E_A(0, y_j, y_k) + q_{Y_j \rightarrow A}(y_j) + q_{Y_k \rightarrow A}(y_k)\} = -0.5$
 $r_{A \rightarrow Y_i}(1) = \max_{y_j, y_k \in \{0,1\}} \{-E_A(1, y_j, y_k) + q_{Y_j \rightarrow A}(y_j) + q_{Y_k \rightarrow A}(y_k)\} = 0.5$

In order to calculate the maximal state y^* we apply back-tracking

1. $y_i^* \in \operatorname{argmax}_{y_i \in \{0,1\}} r_{A \rightarrow Y_i}(y_i) = \{1\}$
2. $y_j^* \in \operatorname{argmax}_{y_j} \max_{y_k, y_l \in \{0,1\}} \{-E_A(1, y_j, y_k) + q_{Y_i \rightarrow A}(1) + q_{Y_k \rightarrow A}(y_k)\} = \{0\}$
3. $y_k^* \in \operatorname{argmax}_{y_k \in \{0,1\}} \{r_{A \rightarrow Y_k}(1, 0, y_k) + r_{B \rightarrow Y_k}(y_k) + r_{C \rightarrow Y_k}(y_k)\}$
 $= \operatorname{argmax}_{y_k \in \{0,1\}} \{-E_A(1, 0, y_k) + r_{B \rightarrow Y_k}(y_k)\} = \{1\}$
4. $y_l^* \in \operatorname{argmax}_{y_l \in \{0,1\}} \{-E_C(y_k, 1) + q_{Y_k \rightarrow C}(1)\} = \{0\}$

Therefore, the optimal state $y^* = (1, 0, 1, 0)$.

Human pose estimation

The model

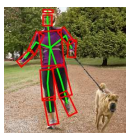
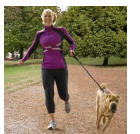
The goal is to recognize an articulated object with joints connecting different parts, here it is a human body.

An object is composed of a number of rigid parts. Each part is modeled as a rectangle parameterized by (x, y, s, θ) , where

- (x, y) means the **center of the rectangle**,
- $s \in [0, 1]$ is a **scaling factor**, and
- the **orientation** is given by θ .

In overall, we have a four-dimensional pose space.

We denote the **locations** of two (connected) parts by $l_i = (x_i, y_i, s_i, \theta_i)$ and $l_j = (x_j, y_j, s_j, \theta_j)$, respectively.

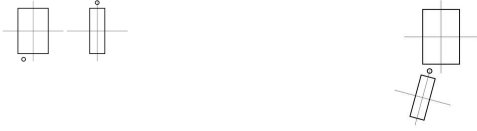


The model (cont.)

Sum-product alg. Max-sum alg. Human pose estimation Loopy belief propagation

An object (e.g., human body) is given by a configuration $L = (l_1, \dots, l_n)$, where l_i specifies the location of part v_i . The connections encode generic relationships such as "close to", "to the left of", or more precise geometrical constraints such as ideal joint angles.

- The **location of a joint** between v_i and v_j is specified by two points (x_{ij}, y_{ij}) and (x_{ji}, y_{ji}) .
- The **relative orientation** is given by θ_{ij} , which is the difference between the orientation of the two parts.



In principle, all parts depend on each other, however, tree structured model can be considered for an articulated pose.

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Graphical representation

Sum-product alg. Max-sum alg. Human pose estimation Loopy belief propagation

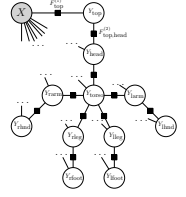
The structure is encoded by a graph $G = (V, E)$, where $V = \{v_1, \dots, v_n\}$ corresponds to n parts, and there is an edge $(v_i, v_j) \in E$ for each pair of connected parts v_i and v_j .

We want to minimize the following energy function

$$L^* \in \operatorname{argmin}_L \left(\sum_{i=1}^n m_i(l_i) + \sum_{(v_i, v_j) \in E} d_{ij}(l_i, l_j) \right),$$

where $m_i(l_i)$ measures the degree of mismatch when the part v_i is placed at location l_i and $d_{ij}(l_i, l_j)$ measures the degree of deformation of the model when part v_i is placed at location l_i and part v_j is placed at location l_j .

Note that MAP inference can be efficiently done by making use of *Max-sum algorithm*.



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Image filters *

Sum-product alg. Max-sum alg. Human pose estimation Loopy belief propagation

The **image filtering** is a technique for modifying or enhancing an image (e.g., smoothing, edge detection, sharpening). For example, the smoothing of an input signal means of removing (or filtering out) high-frequency components.

A **digital image** can be considered as a two dimensional (discretized) signal that is $f: \mathbb{Z}^2 \rightarrow \mathbb{Z}^D$. For example $D = 3$ for color images.

Here we consider **linear filtering** in which the value of an output pixel is a linear combination of the values of the pixels in the input pixel's neighborhood. In a spatially discrete setting, a linear filter is a weighted sum:

$$g(x_0, y_0) = [f * w](x_0, y_0) = \sum_{m,n} w(m, n) f(x_0 - m, y_0 - n)$$

which is also called **discrete convolution** of f and w . In practice this summation extends over a certain neighborhood. The matrix of weights $w(m, n)$ is called a **mask**.

(For more details please refer to the course of **Variational Methods for Computer Vision**.)

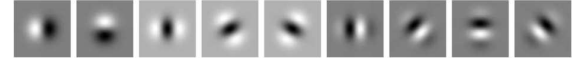
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Unary energies *

Sum-product alg. Max-sum alg. Human pose estimation Loopy belief propagation

An image patch centered at some position is represented by a vector that collects all the responses of a set of Gaussian derivative filters of different orders, orientations and scales at that point. This vector is normalized and called the **iconic index** at that position.



The unary energies are defined as

$$m_i(l_i) = -\ln \mathcal{N}(\alpha(l_i), \mu_i, \Sigma_i),$$

where $\alpha(l_i)$ is the iconic index at location l_i in the image.

The parameters for each part (i.e. the mean vector μ_i and the covariance matrix Σ_i) can be obtained by maximum likelihood estimation for a given set of training samples.

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Pairwise energies *

Sum-product alg. Max-sum alg. Human pose estimation Loopy belief propagation

The pairwise energies have a special form as follows.

$$d_{ij}(l_i, l_j) = -\ln \mathcal{N}(T_{ji}(l_j) - T_{ij}(l_i), \mathbf{0}, \mathbf{D}_{ij}),$$

where where T_{ij} , T_{ji} and \mathbf{D}_{ij} are the connection parameters

$$T_{ij}(l_i) = (x'_i, y'_i, s_i, \cos(\theta_i + \theta_{ij}), \sin(\theta_i + \theta_{ij})),$$

$$T_{ji}(l_j) = (x'_j, y'_j, s_j, \cos(\theta_j), \sin(\theta_j)),$$

$$\mathbf{D}_{ij} = \operatorname{diag}(\sigma_x^2, \sigma_y^2, \sigma_s^2, 1/k, 1/k).$$

$T_{ij}(l_i)$ and $T_{ji}(l_j)$ are one-to-one mappings encoding the set of possible transformed locations.

This special form for the pairwise energies allows for matching algorithms that run in $\mathcal{O}(h')$, where h' is the number of grid locations in a discretization of the space. This results in the time complexity $\mathcal{O}(h'n)$ rather than $\mathcal{O}(h^2n)$.

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Pairwise energies (cont.) *

Sum-product alg. Max-sum alg. Human pose estimation Loopy belief propagation

Let \mathbf{R} be the matrix that performs a rotation of θ radians about the origin. Then,

$$\begin{bmatrix} x'_i \\ y'_i \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} + s_i \mathbf{R}_{\theta_i} \begin{bmatrix} x_{ij} \\ y_{ij} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x'_j \\ y'_j \end{bmatrix} = \begin{bmatrix} x_j \\ y_j \end{bmatrix} + s_j \mathbf{R}_{\theta_j} \begin{bmatrix} x_{ji} \\ y_{ji} \end{bmatrix},$$

where (x_i, y_i) , (x_j, y_j) and (x_{ij}, y_{ij}) , (x_{ji}, y_{ji}) are the positions of the joints in image and local coordinates, respectively.

We assume the following joint distributions:

- $\mathcal{N}(x_i - x_j, \mathbf{0}, \sigma_x^2)$ and $\mathcal{N}(y_i - y_j, \mathbf{0}, \sigma_y^2)$ which measures the horizontal and vertical distances, respectively, between the observed joint positions.
- $\mathcal{N}(s_i - s_j, 0, \sigma_s^2)$ measures the difference in foreshortening between the two parts.
- $\mathcal{M}(\theta_i - \theta_j, \theta_{ij}, k) \propto \exp(k \cos(\theta_i - \theta_j - \theta_{ij}))$ measures the difference between the relative angle of the two parts and the ideal relative angle.

These parameters can be also obtained by maximum likelihood estimation.

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9. Belief Propagation - 30 / 38

Inference

Sum-product alg. Max-sum alg. Human pose estimation Loopy belief propagation

MAP inference provides a single (best) prediction of the overall pose. The factor-to-variable messages can be written as

$$r_{F \rightarrow v_i}(l_i) = \max_{\substack{(l'_i, l'_j) \in \mathcal{Y}_F \\ l'_i = l_i}} \left(\exp(-m_i(l'_i) - d_{ij}(l'_i, l'_j)) + \sum_{k \in N(F) \setminus \{i\}} q_{v_k \rightarrow F}(l'_k) \right).$$

\mathcal{Y} could be quite large ($\approx 1.5M$ possible states), hence $\mathcal{Y}_i \times \mathcal{Y}_j$ is too big. However a special form of pairwise energies is used, so that a message can be calculated in $\mathcal{O}(|\mathcal{Y}_i|)$ time.

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9. Belief Propagation - 31 / 38

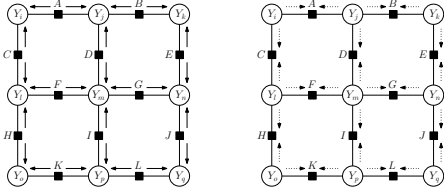
Loopy belief propagation

Sum-product alg. Max-sum alg. Human pose estimation Loopy belief propagation

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9. Belief Propagation - 31 / 38

When the graph has cycles, then there is no well-defined *leaf-to-root* order. However, one can apply message passing on cyclic graphs, which results in **loopy belief propagation**.



1. Initialize all messages as constant 1
2. Pass factor-to-variables and variables-to-factor messages alternately until convergence
3. Upon convergence, treat **beliefs** μ_F as approximate marginals

The factor-to-variable messages $r_{F \rightarrow Y_i}$ remain well-defined and are computed as before.

$$r_{F \rightarrow Y_i}(y_i) = \sum_{\substack{y'_F \in \mathcal{Y}_F \\ y'_i = y_i}} \left(\exp(-E_F(y'_F)) \prod_{j \in N(F) \setminus \{i\}} q_{Y_j \rightarrow F}(y'_j) \right)$$

The variable-to-factor messages are normalized at every iteration as follows:

$$q_{Y_i \rightarrow F}(y_i) = \frac{\prod_{F' \in M(i) \setminus \{F\}} r_{F' \rightarrow Y_i}(y_i)}{\sum_{y_j \in \mathcal{Y}_j} \prod_{F' \in M(j) \setminus \{F\}} r_{F' \rightarrow Y_j}(y_j)}$$

In case of tree structured graphs, in the sum-product algorithm these normalization constants are equal to 1, since the marginal distributions, calculated in each iteration, are exact.

The approximate marginals, i.e. beliefs, are computed as before but now a factor-specific normalization constant z_F is also used.

The factor marginals are given by

$$\mu_F(y_F) = \frac{1}{z_F} \exp(-E_F(y_F)) \prod_{i \in N(F)} q_{Y_i \rightarrow F}(y_i),$$

where the factor specific constant is given by

$$z_F = \sum_{y_F \in \mathcal{Y}_F} \exp(-E_F(y_F)) \prod_{i \in N(F)} q_{Y_i \rightarrow F}(y_i).$$

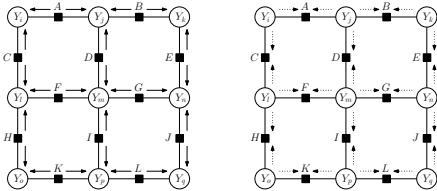
In addition to the factor marginals the algorithm also computes the variable marginals in a similar fashion.

$$\mu_i(y_i) = \frac{1}{z_i} \prod_{F' \in M(i)} r_{F' \rightarrow Y_i}(y_i),$$

where the normalizing constant is given by

$$z_i = \sum_{y_i \in \mathcal{Y}_i} \prod_{F' \in M(i)} r_{F' \rightarrow Y_i}(y_i).$$

Since the local normalization constant z_F differs at each factor for loopy belief propagation, the exact value of the normalizing constant Z cannot be directly calculated. Instead, an approximation to the log partition function can be computed.



Loopy belief propagation is very popular, but has some problems:

- It might not converge (e.g., it can oscillate).
- Even if it does, the computed probabilities are only *approximate*.
- If there is a single cycle only in the graph, then it converges.

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