Combinatorial Optimization in Computer Vision (IN2245)

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10. Tree-reweighted Message Passing & Mean Field Methods

Tree-reweighted message passing

Introduction

Assume an undirected (pairwise) graphical model $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the following energy function:

$$E(\mathbf{y}) = \text{const} + \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j) \ . \qquad (1)$$

For each $i \in \mathcal{V}$, let Y_i be a random variable taking values from a (finite) set $\mathcal{Y}_i$, therefore $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$.

Let us introduce the following notations

- $E_i(a) \triangleq \theta_{i;a}$, which is a vector of size $|\mathcal{Y}_i|$.
- $E_{ij}(a,b) \triangleq \theta_{ij;ab}$, which is a vector of size $|\mathcal{Y}_i \times \mathcal{Y}_j|$. Note that $\theta_{ij;ab} \equiv \theta_{ji;ab}$.

One can consider

$$\theta = \{\theta_\alpha \mid \alpha \in \mathcal{I}\} \in \mathbb{R}^d$$

as a vector, where $\mathcal{I} = \{\text{const}\} \cup \{(i;a)\} \cup \{(ij;ab)\}$.

Canonical overcomplete representation

The energy function (1) can be written (with equivalent notations) as

$$E(\mathbf{y}; \theta) = \theta_{\text{const}} + \sum_{i \in \mathcal{V}} \theta_{i;y_i} + \sum_{(i,j) \in \mathcal{E}} \theta_{ij;y_i y_j} \ .$$

We introduce a mapping $\phi : \mathcal{Y} \to \mathbb{R}^d$ so that

$$E(\mathbf{y}; \theta) = \langle \theta, \phi(\mathbf{y}) \rangle = \sum_{\alpha \in \mathcal{I}} \theta_\alpha \phi_\alpha(\mathbf{y}) \ .$$

The mapping ϕ is called the **canonical overcomplete representation** consists of the following functions $\phi_\alpha : \mathcal{Y} \to \mathbb{R}$:

$$\begin{aligned} \phi_{\text{const}}(\mathbf{y}) &= 1 \\ \phi_{i;a}(\mathbf{y}) &= [\![y_i = a]\!] \\ \phi_{ij;ab}(\mathbf{y}) &= [\![y_i = a, y_j = b]\!] \ . \end{aligned}$$

Revisit the Max-sum algorithm

Reminder: the *Max-sum algorithm* solves the following optimization problem:

$$y^* \in \operatorname*{argmax}_{y \in \mathcal{Y}} p(y) = \operatorname*{argmax}_{y \in \mathcal{Y}} \frac{1}{Z} \exp\left(\sum_{F \in \mathcal{F}} -E_F(y_F) \right) = \operatorname*{argmin}_{y \in \mathcal{Y}} \sum_{F \in \mathcal{F}} E_F(y_F) \ .$$

It maintains messages $M_{ij} = \{M_{ij;a} \mid a \in \mathcal{Y}_j\}$ for each $(i,j) \in \mathcal{E}$, where

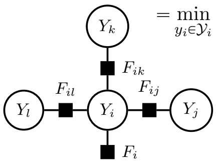
$$M_{ij;b} := \min_{a \in \mathcal{Y}_i} \{(\theta_{i;a} + \sum_{(s,i) \in \mathcal{E}, s \neq j} M_{si;a}) + \theta_{ij;ab}\} + \text{const}_i \ .$$

$M = \{M_{ij}\}$ denotes the vector of all messages.

Revisit the Max-sum algorithm

Assuming the following factor graph, let us calculate the message $r_{F_{ij} \to Y_j}$.

$$\begin{aligned} r_{F_{ij} \to Y_j}(y_j) &= \min_{y_i \in \mathcal{Y}_i} \{E_{ij}(y_i, y_j) + q_{Y_i \to F_{ij}}(y_i)\} \\ &= \min_{y_i \in \mathcal{Y}_i} \left\{ E_{ij}(y_i, y_j) + \sum_{F \in M(i) \setminus \{F_{ij}, F_i\}} r_{F \to Y_i}(y_i) + r_{F_i \to Y_i}(y_i) \right\} \\ &= \min_{y_i \in \mathcal{Y}_i} \left\{ (E_i(y_i) + \sum_{F \in M(i) \setminus \{F_{ij}, F_i\}} r_{F \to Y_i}(y_i)) + E_{ij}(y_i, y_j) \right\} \\ &= \min_{y_i \in \mathcal{Y}_i} \left\{ (\theta_{i;y_i} + \sum_{s \in N(F) \setminus \{i,j\}, F \in M(i)} M_{si;y_i}) + \theta_{ij;y_i y_j} \right\} = M_{ij;y_j} \ . \end{aligned}$$



Reparameterization

Assuming two parameterization θ and $\bar{\theta}$, if they define the same energy function, i.e. $E(\mathbf{y}; \theta) = E(\mathbf{y}; \bar{\theta})$ for all $\mathbf{y} \in \mathcal{Y}$, denoted by $\theta \equiv \bar{\theta}$, then θ is called a **reparameterization** of $\bar{\theta}$.

Note that this condition does not necessarily imply that $\theta = \bar{\theta}$. Indeed, any message vector $M = \{M_{st}\}$ defines reparameterization $\theta = \bar{\theta}[M]$ as follows:

$$\begin{aligned} \theta_i &= \bar{\theta}_i + \sum_{(i,j) \in \mathcal{E}} M_{ij} \\ \theta_{ij;ab} &= \bar{\theta}_{ij;ab} - M_{ij;b} - M_{ji;a} \\ \theta_{\text{const}} &= \bar{\theta}_{\text{const}} \end{aligned}$$

In belief propagation (BP) we can alternatively store the reparameterization $\theta = \bar{\theta}[M]$ instead of $\bar{\theta}$ and M. Namely, sending a message from node i to j is equivalent to reparameterizing vectors θ_i and θ_{ij}.

Normal form

A message for an edge $(i, j) \in \mathcal{E}$ is called **valid** if any update does not change M_{ij}. A message for $(i, j) \in \mathcal{E}$ is valid iff

$$\min_{a \in \mathcal{Y}_i} \{\theta_{i;a} + \theta_{ij;ab}\} = \text{const}_{ij} \quad \forall b \in \mathcal{Y}_j \; .$$

That is a message from s to t does not change θ_{ij} and θ_i. We say that θ is in a **normal form** if all messages are valid.
Minimum value of the energy is given by $\Phi(\theta) = \min_{\mathbf{y} \in \mathcal{Y}} E(\mathbf{y}; \theta)$ and the min-marginals for nodes and edges are given by

$$\Phi_{i;a}(\theta) = \min_{\mathbf{y} \in \mathcal{Y}, y_i = a} E(\mathbf{y}; \theta) \quad \text{and} \quad \Phi_{ij;ab}(\theta) = \min_{\mathbf{y} \in \mathcal{Y}, y_i = a, y_j = b} E(\mathbf{y}; \theta) \; .$$

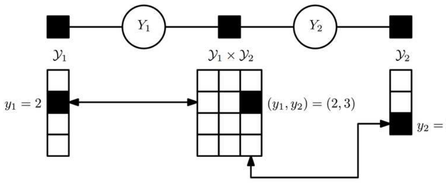
For a tree-structured graph the values $\theta_{i;a}$ and $\theta_{ij;ab}$ for vector θ in a normal correspond to min-marginals (up to a constant):

$$\begin{aligned} \Phi_{i;a}(\theta) &= \theta_{i;a} && + \text{const}_i \\ \Phi_{ij;ab}(\theta) &= \{\theta_{i;a} + \theta_{ij;ab} + \theta_{j;b}\} && + \text{const}_{ij} \end{aligned} \tag{2}$$

LP relaxation

In general, energy minimization (1) is NP-hard. Therefore, one can focused on approximation, such as **linear programming relaxation**. The constraint set is as follows:

$$\mathcal{L}(\mathcal{G}) = \Big\{\tau \in \mathbb{R}^d_+ \mid \tau_{\text{const}} = 1, \sum_{a \in \mathcal{Y}_i} \tau_{i;a} = 1, \sum_{a \in \mathcal{Y}_i} \tau_{ij;ab} = \tau_{j;b}\Big\}$$



The following minimization problem yields a lower bound on $\Phi(\bar{\theta})$:

$$\min_{\tau \in \mathcal{L}(\mathcal{G})} \langle \bar{\theta}, \tau \rangle \; . \tag{3}$$

Convex combinations of trees

We need to introduce some notation. Let $\mathcal{T}$ be a collection of trees in graph $\mathcal{G}$ and $\rho^T > 0$, $T \in \mathcal{T}$ be some distribution on $\mathcal{T}$. We assume that each edge is covered by at least one tree.
For a given tree $T = (\mathcal{V}^T, \mathcal{E}^T)$ we define a set of indices associated with vertices and edges in the tree:

$$\mathcal{I}^T = \{\text{const}\} \cup \{(i; a) \mid i \in \mathcal{V}^T\} \cup \{(ij; ab) \mid (i, j) \in \mathcal{E}^T\} \; .$$

To each tree $T \in \mathcal{T}$, we associate an energy parameter θ^T belonging to the following linear constraint set:

$$\mathcal{A}^T = \{\theta^T \in \mathbb{R}^d \mid \theta^T_\alpha = 0 \; \forall \alpha \in \mathcal{I} \backslash \mathcal{I}^T\} \; .$$

By concatenating all of the tree vectors, we get a vector $\boldsymbol{\theta} = \{\theta^T \mid T \in \mathcal{T}\} \in \mathbb{R}^{d \times |\mathcal{T}|}$ belonging to the constraint set

$$\mathcal{A} = \{\boldsymbol{\theta} \in \mathbb{R}^{d \times |\mathcal{T}|} \mid \theta^T \in \mathcal{A}^T \text{ for all } T \in \mathcal{T}\} \; .$$

Convex combinations of trees

Consider function $\Phi_\rho : \mathcal{A} \to \mathbb{R}$ defined as follows:

$$\Phi_\rho(\boldsymbol{\theta}) = \sum_T \rho^T \Phi(\theta^T) = \sum_T \rho^T \min_{\mathbf{y} \in \mathcal{Y}} \langle \theta^T, \phi(\mathbf{y}) \rangle \; .$$

Let $\bar{\theta} = \sum_T \rho^T \theta^T$, then

$$\begin{aligned} \Phi_\rho(\boldsymbol{\theta}) &= \sum_T \rho^T \Phi(\theta^T) = \mathbb{E}[\Phi(\theta^T)] \leqslant \Phi(\mathbb{E}[\theta^T]) = \min_{\mathbf{y} \in \mathcal{Y}} \langle \mathbb{E}[\theta^T], \phi(\mathbf{y}) \rangle \\ &= \min_{\mathbf{y} \in \mathcal{Y}} \Big\langle \sum_T \rho^T \theta^T, \phi(\mathbf{y}) \Big\rangle = \min_{\mathbf{y} \in \mathcal{Y}} \langle \bar{\theta}, \phi(\mathbf{y}) \rangle = \Phi(\bar{\theta}) \; . \end{aligned}$$

To get the tightest bound we can consider the following maximization problem:

$$\max_{\boldsymbol{\theta} \in \mathcal{A}, \sum_T \rho^T \theta^T = \bar{\theta}} \Phi_\rho(\boldsymbol{\theta}) \; . \tag{4}$$

New Tree-reweighing message passing

Theorem 1. *Minimization problem (3) is the dual to maximization problem (4). Strong duality holds, so their optimal values coincide.*

$$\min_{\tau \in \mathcal{L}(\mathcal{G})} \langle \bar{\theta}, \tau \rangle \qquad \leftrightarrow \qquad \max_{\boldsymbol{\theta} \in \mathcal{A}, \sum_T \rho^T \theta^T = \bar{\theta}} \Phi_\rho(\boldsymbol{\theta})$$

The maximization problem (4) is modified by replacing the constraint as

$$\max_{\boldsymbol{\theta} \in \mathcal{A}, \sum_T \rho^T \theta^T \equiv \bar{\theta}} \Phi_\rho(\boldsymbol{\theta}) \; . \tag{5}$$

Theorem 2. *The optimal value of problem (5) equals to the optimal value of problem (4).*

The goal of the reparameterization step is to make sure that the algorithm satisfies the min-marginal property.

Weak tree agreement

Let $\text{OPT}^T(\theta^T)$ be the set of optimal configurations for parameter θ^T and $\text{OPT}(\boldsymbol{\theta}) = \{\text{OPT}^T(\theta^T) \mid T \in \mathcal{T}\} \in (2^{\mathcal{Y}})^{|\mathcal{T}|}$. For two collections $\mathbb{S}, \tilde{\mathbb{S}} \in (2^{\mathcal{Y}})^{|\mathcal{T}|}$, we write $\mathbb{S} \subseteq \tilde{\mathbb{S}}$ if $\mathbb{S}^T \subseteq \tilde{\mathbb{S}}^T$ for every tree T.

$\mathbb{S}$ is **consistent** if it satisfies the following three conditions:

1. For every tree T set $\mathbb{S}^T$ is non-empty.
2. If node i is contained in trees T and T', then for all $\mathbf{y} \in \mathbb{S}^T$ there exists configuration $\mathbf{y}' \in \mathbb{S}^{T'}$ which agrees with $\mathbf{y}$ on node i, i.e. $y_i = y'_i$.
3. If edge (i, j) is contained in trees T and T', then for all $\mathbf{y} \in \mathbb{S}^T$ there exists configuration $\mathbf{y}' \in \mathbb{S}^{T'}$ which agrees with $\mathbf{y}$ on nodes i and j, i.e. $y_i = y'_i, y_j = y'_j$.

Vector $\boldsymbol{\theta} = \{\theta^T\} \in \mathcal{A}$ is said to satisfy the **weak tree agreement condition** if there exists collection $\mathbb{S} \subseteq \text{OPT}(\boldsymbol{\theta})$ which is consistent.

If a vector $\boldsymbol{\theta}$ satisfies the WTA condition, then the TRW-S algorithm will not make any progress, i.e. it will not increase function Φ_ρ.

TRW-S algorithm

0. Initialize $\boldsymbol{\theta}$ so that $\boldsymbol{\theta} \in \mathcal{A}$ and $\sum_T \rho^T \theta^T \equiv \bar{\theta}$.
1. Select some order for nodes and edges in $\mathcal{V} \cup \mathcal{E}$. For each element $\omega \in \mathcal{V} \cup \mathcal{E}$ find all trees $\mathcal{T}_\omega \subseteq \mathcal{T}$ containing ω. If there is more than one tree, then do the following:
   (a) For all trees $T \in \mathcal{T}_\omega$ reparameterize θ^T such that values $\theta^T_{i;a}$ (if $\omega = i$ is a node) or $\theta^T_{i;a} + \theta^T_{ij;ab} \theta^T_{j;b}$ (if $\omega = (i, j)$ is an edge) give correct min-marginals for tree T.
   (b) "Averaging operation":
   If $\omega = i$ is a node then set $\theta^T_i := \frac{1}{\rho_i} \sum_{T \in \mathcal{T}_i} \rho^T \theta^T_i$ for trees $T \in \mathcal{T}_i$
   If $\omega = (i, j)$ is an edge then set $\theta^T_i, \theta^T_{ij}, \theta^T_j$ for trees $T \in \mathcal{T}_{ij}$ so that

$$(\theta^T_{i;a} + \theta^T_{ij;ab} + \theta^T_{j;b}) = \frac{1}{\rho_{ij}} \sum_{T \in \mathcal{T}_{ij}} (\theta^T_{i;a} + \theta^T_{ij;ab} + \theta^T_{j;b})$$

2. Check whether a stopping criterion is satisfied; if yes, terminate, otherwise go to step 1.

Mean Field methods

KL divergence

Assume two discrete probability distributions P and Q. One way to measure the *difference* between P and Q is to calculate the **Kullback–Leibler (KL) divergence** (a.k.a. relative entropy) defined as

$$\begin{aligned} D_{\mathrm{KL}}(P\|Q) = \sum_i P(i)\log\frac{P(i)}{Q(i)} &= \sum_i P(i)\log P(i) - \sum_i P(i)\log Q(i) \\ &= \mathbb{E}_P[\log P(i)] - \mathbb{E}_P[\log Q(i)] \,. \end{aligned}$$

It is defined iff $Q(i)=0$ implies $P(i)=0$, for all i. If $P(i)=0$, then the ith term is interpreted as 0. The KL divergence is always non-negative, moreover $D_{\mathrm{KL}}(P\|Q)=0$ iff $P=Q$ *almost everywhere.*

Interpretation (Information Theory): it is the amount of information lost when Q is used to approximate P. It measures the expected number of extra bits required to code samples from P using a code optimized for Q rather than the code optimized for P.

Motivation

For general (discrete) factor graph models, performing *probabilistic inference* is hard. Assume we are given an **intractable** distribution $p(y \mid x)$. We consider an **approximate distribution** $q(y)$, which is tractable, for $p(y \mid x)$.

One way of finding the best approximating distribution is to pose it as an **optimization problem** over probability distributions: given a distribution $p(y \mid x)$ and a family Q of tractable distributions $q \in Q$ on $\mathcal{Y}$, we want to solve

$$\begin{aligned} q^* \in \operatorname*{argmin}_{q\in Q} D_{\mathrm{KL}}(q(y)\|p(y\mid x)) &= \operatorname*{argmin}_{q\in Q} \sum_{y\in\mathcal{Y}} q(y)\log\frac{q(y)}{p(y\mid x)} \\ &= \operatorname*{argmin}_{q\in Q} \Bigg\{ \underbrace{\sum_{y\in\mathcal{Y}} q(y)\log q(y)}_{-H(q)} - \sum_{y\in\mathcal{Y}} q(y)\log p(y\mid x) \Bigg\} \,. \end{aligned}$$

The term $-\sum_{y\in\mathcal{Y}} q(y)\log q(y) \triangleq H(q)$ is called the **entropy** of the distribution q.

Mean Field methods

$$\begin{aligned} D_{\mathrm{KL}}(q(y)\|p(y\mid x)) &= -H(q) - \sum_{y\in\mathcal{Y}} q(y)\log p(y\mid x) \\ &= -H(q) - \sum_{y\in\mathcal{Y}} q(y)\log\frac{1}{Z(x)}\prod_{F\in\mathcal{F}}\exp(-E_F(y_F;x_F)) \\ &= -H(q) + \sum_{y\in\mathcal{Y}} q(y)\sum_{F\in\mathcal{F}} E_F(y_F;x_F) + \log Z(x) \\ &= -H(q) + \sum_{F\in\mathcal{F}}\sum_{y_F\in\mathcal{Y}_F} \underbrace{\sum_{\substack{y'\in\mathcal{Y},\\ y'_F=y_F}} q(y)}_{\mu_{F,y_F}(q)} E_F(y_F;x_F) + \log Z(x) \\ &= -H(q) + \sum_{F\in\mathcal{F}}\sum_{y_F\in\mathcal{Y}_F} \mu_{F,y_F}(q)E_F(y_F;x_F) + \log Z(x) \,, \end{aligned}$$

where $\mu_{F,y_F}(q) = \sum_{y'\in\mathcal{Y},y'_F=y_F} q(y)$ are the marginals of q.

Gibbs inequality *

If the set Q is rich enough to contain a close approximation to $p(y \mid x)$ and we succeed at finding it, then the marginals of $q*$ will provide a good approximation to the true marginals of $p(y \mid x)$ that are intractable to compute.

Gibbs inequality provides a lower bound on the log *partition function.*

$$0 \leqslant D_{\mathrm{KL}}(q(y)\|p(y\mid x)) = -H(q) + \sum_{F\in\mathcal{F}}\sum_{y_F\in\mathcal{Y}_F} \mu_{F,y_F}(q)E_F(y_F;x_F) + \log Z(x)$$

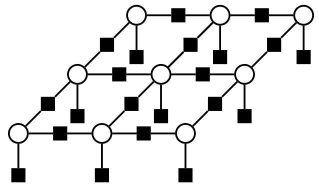
$$\log Z(x) \geqslant H(q) - \sum_{F\in\mathcal{F}}\sum_{y_F\in\mathcal{Y}_F} \mu_{F,y_F}(q)E_F(y_F;x_F) \,.$$

Naive mean field

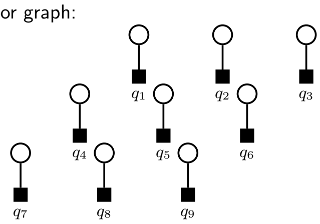
Take a set Q as the set of all distributions in the form:

$$q(y) = \prod_{i\in\mathcal{V}} q_i(y_i) \,.$$

For example, in case of the following factor graph:



Original factor graph



Mean field approximation

Naive mean field

Set Q consists of all distributions in the form:

$$q(y) = \prod_{i\in\mathcal{V}} q_i(y_i) \,.$$

Marginals μ_{F,y_F} take the form

$$\mu_{F,y_F}(q) = \sum_{\substack{y'\in\mathcal{Y},\\ y'_F=y_F}} q(y) = q_{N(F)}(y_F) = \prod_{i\in N(F)} q_i(y_i) \,.$$

Entropy $H(q)$ decomposes as

$$H(q) = \sum_{i\in\mathcal{V}} H_i(q_i) = -\sum_{i\in\mathcal{V}}\sum_{y_i\in\mathcal{Y}_i} q_i(y_i)\log q_i(y_i) \,.$$

Proof. Exercise. □

Naive Mean Field

Putting it together,

$$\begin{aligned} q^* &\in \operatorname*{argmin}_{q\in Q} D_{\mathrm{KL}}(q(y)\|p(y\mid x)) \\ &= \operatorname*{argmin}_{q\in Q} \Bigg\{ -H(q) + \sum_{F\in\mathcal{F}}\sum_{y_F\in\mathcal{Y}_F} \mu_{F,y_F}(q)E_F(y_F;x_F) + \log Z(x) \Bigg\} \\ &= \operatorname*{argmax}_{q\in Q} \Bigg\{ H(q) - \sum_{F\in\mathcal{F}}\sum_{y_F\in\mathcal{Y}_F} \mu_{F,y_F}(q)E_F(y_F;x_F) \Bigg\} \\ &= \operatorname*{argmax}_{q\in Q} \Bigg\{ -\sum_{i\in\mathcal{V}}\sum_{y_i\in\mathcal{Y}_i} q_i(y_i)\log q_i(y_i) - \sum_{F\in\mathcal{F}}\sum_{y_F\in\mathcal{Y}_F} \Big(\prod_{i\in N(F)} q_i(y_i)\Big) E_F(y_F;x_F) \Bigg\} \,. \end{aligned}$$

Optimizing over Q means to optimize over all q_i such that $q_i(y_i) \geqslant 0$ and $\sum_{y_i\in\mathcal{Y}_i} q_i(y_i) = 1$ for all $i \in \mathcal{V}$.

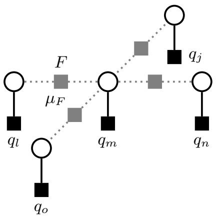
Optimization

$$\operatorname*{argmax}_{q\in Q} \Bigg\{ -\sum_{i\in\mathcal{V}}\sum_{y_i\in\mathcal{Y}_i} q_i(y_i)\log q_i(y_i) - \sum_{F\in\mathcal{F}}\sum_{y_F\in\mathcal{Y}_F} \Big(\prod_{i\in N(F)} q_i(y_i)\Big) E_F(y_F;x_F) \Bigg\} \,.$$

The entropy term is concave and the second term is non-concave due to products of variables occurring in the expression. Therefore solving this non-concave maximization problem globally is hard in general.

Remedy: **block coordinate ascent**

We hold all variables fixed except for a single block q_m, then we obtain a tractable concave maximization problem
→ closed-form update for each q_m.



Lagrange multipliers

To obtain closed form solution, we define the *Lagrangian function*:

$$L(q_i,\lambda) = \Bigg\{ -\sum_{i\in\mathcal{V}}\sum_{y_i\in\mathcal{Y}_i} q_i(y_i)\log q_i(y_i)$$
$$-\sum_{F\in\mathcal{F}}\sum_{y_F\in\mathcal{Y}_F}\Big(\prod_{i\in N(F)} q_i(y_i)\Big)E_F(y_F;x_F) + \lambda\Big(\sum_{y_i\in\mathcal{Y}_i} q_i(y_i) - 1\Big)\Bigg\}.$$

Setting the derivatives of L w.r.t. q_i to 0, we obtain

$$\frac{\partial L}{\partial q_i(y_i)} = 0 = -(\log q_i(y_i)+1) - \sum_{F\in M(i)}\sum_{\substack{y'_F\in\mathcal{Y}_F,\\ y'_i=y_i}}\Big(\prod_{j\in N(F)\setminus\{i\}}\hat{q}_j(y_j)\Big)E_F(y_F;x_F) + \lambda$$

$$q_i^*(y_i) = \exp\Bigg(-1 - \sum_{F\in M(i)}\sum_{\substack{y'_F\in\mathcal{Y}_F,\\ y'_i=y_i}}\Big(\prod_{j\in N(F)\setminus\{i\}}\hat{q}_j(y_j)\Big)E_F(y_F;x_F) + \lambda\Bigg).$$

Lagrange multipliers

λ can be calculated as follows.

$$\sum_{y_i\in\mathcal{Y}_i} q_i(y_i) = \sum_{y_i\in\mathcal{Y}_i}\exp\Bigg(-1 - \sum_{F\in M(i)}\sum_{\substack{y'_F\in\mathcal{Y}_F,\\ y'_i=y_i}}\Big(\prod_{j\in N(F)\setminus\{i\}}\hat{q}_j(y_j)\Big)E_F(y_F;x_F) + \lambda\Bigg)$$

$$\exp(1-\lambda) = \underbrace{\sum_{y_i\in\mathcal{Y}_i}\exp\Bigg(-\sum_{F\in M(i)}\sum_{y'_F\in\mathcal{Y}_F, y'_i=y_i}\Big(\prod_{j\in N(F)\setminus\{i\}}\hat{q}_j(y_j)\Big)E_F(y_F;x_F)\Bigg)}_{Z_i(x_F)}$$

$$\lambda - 1 = -\log Z_i(x_F)\ ,$$

where $Z_i(x_F)$ is a normalizing constant for q_i.

Update equation

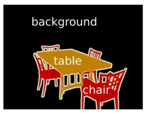
By substituting, we obtain the obtain equation for the Naive Mean Field method

$$q_i^*(y_i) = \exp\Bigg(-\sum_{F\in M(i)}\sum_{\substack{y'_F\in\mathcal{Y}_F,\\ y'_i=y_i}}\Big(\prod_{j\in N(F)\setminus\{i\}}\hat{q}_j(y_j)\Big)E_F(y_F;x_F) - \log Z_i(x_F)\Bigg)$$
$$= \frac{1}{Z_i(x_F)}\exp\Bigg(-\sum_{F\in M(i)}\sum_{\substack{y'_F\in\mathcal{Y}_F,\\ y'_i=y_i}}\Big(\prod_{j\in N(F)\setminus\{i\}}\hat{q}_j(y_j)\Big)E_F(y_F;x_F)\Bigg).$$

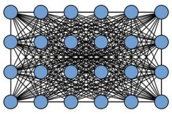
Semantic segmentation

Krähenbühl and Koltun proposed an efficient approximate inference in fully connected CRF model by applying *Naive Mean Field* approach.

Semantic segmentation: assign a label from the set of labels $\mathcal{L} = \{l_1, l_2, \ldots, l_k\}$ for each pixel on the image regarding their semantic meaning.



For each pixel on the image a random variable is assigned taking a value from $\mathcal{L}$. A fully connected pairwise CRF model $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is considered, where the corresponding energy function is given by



$$E(\mathbf{y}) = \sum_{i\in\mathcal{V}} E_i(y_i) + \sum_{(i,j)\in\mathcal{E}} E_{ij}(y_i,y_j)\ ,$$

where $\mathcal{E} = \{(i,j)\in\mathcal{V}\times\mathcal{V} \mid i<j\}$.

Energy functions

- **Unary energies** $E_i(y_i)$ are computed independently for each pixel as $E_i(y_i) = -\log P_i(y_i)$ measures the degree of disagreement between labelling y_i and the image at pixel i.
- **Pairwise energies** (so-called **contrast sensitive Potts-model**), measuring the extent to which the labelling y is not piecewise smooth, have the form (p_i and I_i denote the pixel coordinates and intensity, resp.)

$$E_{ij}(y_i,y_j) = [\![y_i\neq y_j]\!]\sum_m w^{(m)}k^{(m)}(\mathbf{f}_i,\mathbf{f}_j)$$
$$= [\![y_i\neq y_j]\!]\sum_m w^{(m)}\exp\Big(-\frac{1}{2}(\mathbf{f}_i-\mathbf{f}_j)^T\mathbf{\Lambda}^{(m)}(\mathbf{f}_i-\mathbf{f}_j)\Big)$$
$$= [\![y_i\neq y_j]\!]\Big\{w^{(1)}\exp\Big(-\frac{|p_i-p_j|^2}{2\theta_\alpha^2} - \frac{|I_i-I_j|^2}{2\theta_\beta^2}\Big)$$
$$+ w^{(2)}\exp\Big(-\frac{|p_i-p_j|^2}{2\theta_\gamma^2}\Big)\Big\}.$$

The parameters $\theta_\alpha, \theta_\beta$ and θ_γ are estimated on a set of training images.

Inference

The inference is based on Naive Mean Field approximation, where the update equation is given by

$$q_i(y_i) = \frac{1}{Z_i}\exp\Bigg\{-E_i(y_i) - \sum_{l'\in\mathcal{L}}[\![y_i\neq y_j]\!]\sum_{m=1}^{K} w^{(m)}\sum_{i\neq j}k^{(m)}(\mathbf{f}_i,\mathbf{f}_j)q_j(l')\Bigg\}\ .$$

The inference is performed in average 0.2 seconds for 500.000 variables (in contrast to 36 hours).
The main idea: the message passing step can be expressed as a convolution with a Gaussian kernel $G_{\mathbf{\Lambda}^{(m)}}$ in feature space:

$$\sum_{j\in\mathcal{V}} k^{(m)}(\mathbf{f}_i,\mathbf{f}_j)q_j(l) - q_i(l) = [G_{\mathbf{\Lambda}^{(m)}} * q(l)](\mathbf{f}_i) - q_i(l)\ .$$

Note that the convolution sums over all variables, while message passing does not sum over q_i. This convolution can be efficiently calculated in $\mathcal{O}(|\mathcal{V}|)$ time (instead of $\mathcal{O}(|\mathcal{V}|^2)$).

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