Combinatorial Optimization in Computer Vision (IN2245)

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14. Higher-order Clique Reduction

Higher-order energy functions

So far, we have considered (undirected) **pairwise** graphical models $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the corresponding energy function is defined for a labeling $\mathbf{y} \in \mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_{|\mathcal{V}|}$ as

$$E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j) .$$

The pairwise model suffers from a number of problems stemming from its inability to express high-order dependencies between pixels.

In many computer vision problems, however, one needs to use higher-order relations of the pixels.

$$E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j) + \sum_{c \in \mathcal{C}} E_c(y_c) ,$$

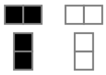
where $\mathcal{C}$ is a set of cliques with at least three variables and $y_c \in \times_{i \in c} \mathcal{Y}_i$.

Image denoising

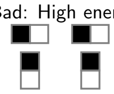
Original (binary) | Noise-added | Denoised

Pairwise interactions:

Good: Low energy

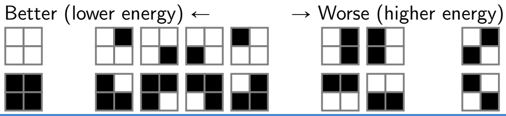


Bad: High energy



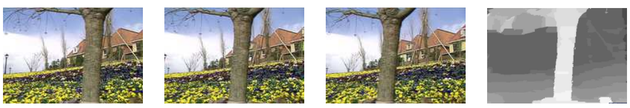
Higher-order interactions:

Better (lower energy) ← → Worse (higher energy)



Multi-camera scene reconstruction

This problem is the generalization of *stereo matching* (see Lecture 12).



Left view | Middle view | Right view | Reconstruction

We are given three images I_l, I_m and I_r. For each pixel p, on all the three images, we consider a label y_p corresponding to a (discretized) depth value. Note that a pair (p, y_p) specifies a 3D point that has the depth value y_p and is projected to the pixel p.

Let $\mathcal{I}$ be a set of triples of "nearby" 3D points: these points will come from different cameras, but they will share the same depth (i.e. the points are of the form (p, y_p), (q, y_q) and (r, y_r), where $y_p = y_q = y_r$ and p, q and r are pixels from different cameras).

Multi-camera scene reconstruction

If three pixels have similar intensities, then it is more likely that they see the same scene element than if only two pixels have similar intensities.

The energy function is defined as

$$E(\mathbf{y}) = \sum_{((p,y_p),(q,y_q),(r,y_r)) \in \mathcal{I}} E^l_{pqr}(y_p, y_q, y_r) + \sum_{\alpha \in \{l,m,r\}} \sum_{(i,j) \in \mathcal{N}_\alpha} [\![y_i \neq y_j]\!] ,$$

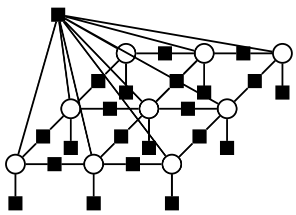
where $\mathcal{N}_l, \mathcal{N}_m$ and $\mathcal{N}_r$ are neighborhood systems on pixels corresponding to single camera images,

$$E^l_{pqr}(y_p, y_q, y_r) = D(p, q, r) \cdot [\![y_p = y_q = y_r]\!] .$$

The data term enforces photoconsistency, so $D(p, q, r)$ is a function of the intensity difference between p, q and r.

Semantic segmentation

One may want to enforce label consistency by applying higher-order terms.



Original | Pairwise CRF | Higher-order CRF

Reduction by substitution

Multi-linear representation

Reminder: any pseudo–Boolean function $f : \mathbb{B}^n \to \mathbb{R}$ can be uniquely written as a *multi-linear function* (see Lecture 2.):

$$f(x_1, \ldots, x_n) = \sum_{S \subset \mathcal{V}} c_S \prod_{i \in S} x_i \ ,$$

where $\mathcal{V} = \{1, \ldots, n\}$ and $c_S \in \mathbb{R}$.

Remark that for a pseudo–Boolean function f given in the above multi-linear form the following holds:

$$\max_{x_1,\ldots,x_n \in \mathbb{B}} f(x_1, \ldots, x_n) \leqslant \sum_{S \subset \mathcal{V}} |c_S| \ .$$

Reduction by substitution

Rosenberg proposed a method that reduces the minimization of a pseudo–Boolean function of **any degree** to an equivalent problem for **quadratic** pseudo–Boolean function.

Idea: the **product of two variables** xy **is replaced by a new variable** z, which is forced to have the same value as xy at any minimum of the function.

Assume that $x, y, z \in \mathbb{B}$ and define

$$g(x, y, z) = xy - 2xz - 2yz + 3z \ .$$

Then the following equivalences hold:

$$xy = z \quad \Leftrightarrow \quad g(x, y, z) = 0 \quad \text{and}$$
$$xy \neq z \quad \Leftrightarrow \quad g(x, y, z) > 0 \ .$$

Note that, of course, the above quadratic expression is not the only one for which such equivalences would hold.

Reduction by substitution

x	y	z	$xy - 2xz - 2yz + 3z$	$xy = z$
0	0	0	0	yes
0	0	1	3	no
0	1	0	0	yes
0	1	1	1	no
1	0	0	0	yes
1	0	1	1	no
1	1	0	1	no
1	1	1	0	yes

$xy = z \Leftrightarrow g(x, y, z) = 0$ and
$xy \neq z \Leftrightarrow g(x, y, z) > 0$

Consider an *example* pseudo–Boolean function

$$f(x, y, w) = xyw + xy + y \ .$$

Then replace xy by z and add $Mg(x, y, z)$:

$$\tilde{f}(x, y, w, z) = zw + z + y + Mg(x, y, z) \quad \text{for} \quad 0 < M \in \mathbb{R} \ .$$

Reduction by substitution

$$f(x, y, w) = xyw + xy + y$$
$$\tilde{f}(x, y, w, z) = zw + z + y + Mg(x, y, z) \ .$$

Let us define M as $M := 1 + \sum_{S \in \mathcal{V}} |c_S| = 1 + 3 = 4$.

- If $xy = z$, then $g(x, y, z) = 0$, thus $\min \tilde{f}(x, y, w, z) = \min f(x, y, w)$.
- If $xy \neq z$, then $\tilde{f}(x, y, w, z) \geqslant M$. Nevertheless, $\max f(x, y, w) < M$, therefore it is impossible for $\tilde{f}$ to take the minimum whenever $xy \neq z$.

By repeating the above reduction, any higher-order function can be reduced to a quadratic one with additional variables: $\tilde{f}$ has **one more variable** and is of **one less degree** than the original function f.

For any minimum-energy value-assignment for the new function, the same assignment of values to the original variables gives the minimum energy to the original function:

$$(x^*, y^*, w^*, z^*) \in \operatorname*{argmin}_{x,y,w,z \in \mathbb{B}} \tilde{f}(x, y, w, z) \quad \Rightarrow \quad f(x^*, y^*, w^*) = \min_{x,y,w \in \mathbb{B}} f(x, y, w) \ .$$

The problem with reduction by substitution

The first term in

$$Mg(x, y, z) = Mxy - 2Mxz - 2Myz + 3Mz$$

is a quadratic term with a (very) large positive coefficient. According to Theorem 4 in Lecture 2, this makes in all cases the result of reduction non-submodular. It seems such an energy cannot be minimized very well even with QPBO method (see Lecture 8).

Example:

$$f(x_1, x_2, x_3, x_4) = -2x_1x_2 - 3\underbrace{x_2x_3}_{x_5}x_4$$

$$\begin{aligned}\tilde{f}(x_1, \ldots, x_5) &= -2x_1x_2 - 3x_5x_4 + Mg(x_2, x_3, x_5) \\ &= -2x_1x_2 - 3x_4x_5 + 6(x_2x_3 - 2x_2x_5 - 2x_3x_5 + 3x_5) \\ &= -2x_1x_2 + 6x_2x_3 - 12x_2x_5 - 12x_3x_5 - 3x_4x_5 + 18x_5 \ .\end{aligned}$$

Reduction by minimum selection

Reduction by minimum selection

Consider a **cubic** pseudo–Boolean function of $x, y, z \in \mathbb{B}$ for $a \in \mathbb{R}$

$$f(x, y, z) = axyz \ .$$

Observe that

$$xyz = \max_{w \in \mathbb{B}} w(x + y + z - 2) \ .$$

x	y	z	xyz	$x + y + z - 2$	$\max_{w \in \mathbb{B}} w(x + y + z - 2)$
0	0	0	0	-2	0
0	0	1	0	-1	0
0	1	0	0	-1	0
0	1	1	0	0	0
1	0	0	0	-1	0
1	0	1	0	0	0
1	1	0	0	0	0
1	1	1	1	1	1

Reduction by minimum selection

Suppose $a < 0$,

$$axyz = a \max_{w \in \mathbb{B}} w(x + y + z - 2) = \min_{w \in \mathbb{B}} aw(x + y + z - 2) \ .$$

Suppose $a > 0$, and let $\bar{x} = 1 - x$, $\bar{y} = 1 - y$ and $\bar{z} = 1 - z$, then

$$\begin{aligned}\bar{x}\bar{y}\bar{z} &= \max_{w \in \mathbb{B}} w(\bar{x} + \bar{y} + \bar{z} - 2) = \max_{w \in \mathbb{B}} w(1 - x + 1 - y + 1 - z - 2) \\ &= -\min_{w \in \mathbb{B}} w(x + y + z - 1) \ .\end{aligned}$$

Furthermore,

$$\bar{x}\bar{y}\bar{z} = (1 - x)(1 - y)(1 - z) = -(x + y + z) + (xy + xz + yz) - xyz + 1 \ .$$

Therefore, we obtain

$$axyz = \min_{w \in \mathbb{B}} a\Big(w(x + y + z - 1) - (x + y + z) + (xy + xz + yz) + 1\Big) \ .$$

The cubic case

Thus, either case, the cubic term can be replaced by quadratic terms. When $axyz$ appears in a minimization problem with

- $a < 0$, it can be replaced by

$$aw(x + y + z - 2) = a(wx + wy + wz - 2w) \,.$$

- $a > 0$, it can be replaced by

$$\begin{aligned} &a\big(w(x + y + z - 1) - (x + y + z) + (xy + yz + zx) + 1\big) \\ &\quad = a(wx + wy + wz + xy + yz + zx - x - y - z - w + 1) \,. \end{aligned}$$

Note that this reduction is valid either the function is submodular or not.

The quartic case

For quartic term $axyzt$ $(x, y, z, t \in \mathbb{B})$, the same trick works if $a < 0$, that is

$$axyzt = \min_{w\in\mathbb{B}} aw(x + y + z + t - 3) \,.$$

However, if $a > 0$,

$$\begin{aligned} \bar{x}\bar{y}\bar{z}\bar{t} &= \max_{w\in\mathbb{B}} w(\bar{x} + \bar{y} + \bar{z} + \bar{t} - 3) \\ &= \max_{w\in\mathbb{B}} w(1-x+1-y+1-z+1-t-3) = \max_{w\in\mathbb{B}} w(-x - y - z - t + 1) \,. \end{aligned}$$

Furthermore,

$$\begin{aligned} \bar{x}\bar{y}\bar{z}\bar{t} = &(1 - x)(1 - y)(1 - z)(1 - t) \\ = &xyzt - (xyz + xyt + xzt + yzt) + (xy + xz + xt + yz + yt + zt) \\ &- (x + y + z + t) + 1 \,. \end{aligned}$$

The quartic case

Therefore, we obtain

$$\begin{aligned} \bar{x}\bar{y}\bar{z}\bar{t} = &\max_{w\in\mathbb{B}} w(-x - y - z - t + 1) + (xyz + xyt + xzt + yzt) \\ &- (xy + xz + xt + yz + yt + zt) + (x + y + z + t) - 1 \,. \end{aligned}$$

Unlike the cubic case, the maximization problem is not turned into a minimization. Similarly, this does not work with any term of even degree.

In general, for negative higher-order terms the following holds:

$$-x_1 \cdots x_d = \min_{w\in\mathbb{B}} w\left((d - 1) - \sum_{i=1}^{d} x_i\right) \,.$$

Symmetric polynomials

A polynomial $p(x_1, x_2, \ldots, x_n)$ is said to be **symmetric polynomial**, if for any permutation π of the subscripts $1, 2, \ldots, n$ the following holds

$$p(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}) = p(x_1, x_2, \ldots, x_n) \,.$$

That is interchanging any of the variables, one obtain the same polynomial.

Elementary symmetric polynomials

The **elementary symmetric polynomials** in n variables for $k = 0, 1, \ldots, n$, are defined by

$$\begin{aligned} \sigma_0(x_1, x_2, \ldots, x_n) &= 1 \,, \\ \sigma_1(x_1, x_2, \ldots, x_n) &= \sum_{1\leqslant i\leqslant n} x_i \,, \\ \sigma_2(x_1, x_2, \ldots, x_n) &= \sum_{1\leqslant i<j\leqslant n} x_i x_j \,, \end{aligned}$$

and so on, ending with

$$\sigma_n(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n \,.$$

It is known that any symmetric polynomial can be expressed as a polynomial in elementary symmetric polynomials. That is, any symmetric polynomial is given by an expression involving only additions and multiplication of constants and elementary symmetric polynomials.

The quartic and quintic cases

We would like to generalize the formulæ shown before. Observe that the equations in the cubic case are **symmetric** in the three variables x, y and z.

Therefore, if there exists a generalization for $xyzt$, the RHS should also be symmetric in the four variables:

$$xyzt = \min_{w\in\mathbb{B}} w(\text{1st order sym.}) + (\text{2nd order sym.}) \,.$$

We will write symmetric polynomials as a polynomial of elementary symmetric polynomials. There is only one elementary symmetric polynomial of each degree; the ones we need are:

$$s_0 = 1 \,, \quad s_1 = x + y + z + t \,, \quad s_2 = xy + yz + zx + tx + ty + tz \,.$$

Since $x, y, z, t \in \mathbb{B}$,

$$s_1^2 = (x+y+z+t)^2 = x+y+z+t+2xy+2yz+2zx+2tx+2ty+2tz = s_1+2s_2 \,.$$

The quartic and quintic cases

Therefore, any symmetric polynomial up to second degree, with binary variables, can be written as a linear combination of s_0, s_1 and s_2. Thus the formula should be of the form for $a, b, c, d, e \in \mathbb{Z}$

$$xyzt = \min_{w\in\mathbb{R}} w(as_1 + b) + (cs_2 + ds_1 + es_0) \,.$$

By applying **exhaustive search** for a, b, c, d and e one can obtain that

$$xyzt = \min_{w\in\mathbb{B}} w(2s_1 + 3) + s_2 \,.$$

One can similarly obtain formula for **quintic case**. However, the number of auxiliary variables **increases**. That is

$$xyztu = \min_{(v,w)\in\mathbb{B}} (v(-2r_1 + 3) + w(-r1 + 3)) + r_2 \,,$$

where r_1 and r_2 are the 1st and 2nd degree elementary symmetric polynomials in x, y, z, t, u.

Even degree *

Theorem 1. *Let $x_1, \ldots, x_d \in \mathbb{B}$, where d is even. For $a > 0$*

$$ax_1 \cdots x_d = \min_{w_1,\ldots,w_{n_d}\in\mathbb{B}} a\left(\sum_{i=1}^{n_d} w_i(-2\sigma_1 + 4i - 1)\right) + a\sigma_2 \,,$$

where

$$n_d = \left\lfloor \frac{d-1}{2} \right\rfloor \,, \quad \sigma_1 = \sum_{i=1}^{d} x_i \,, \quad \text{and} \quad \sigma_2 = \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} x_i x_j \,.$$

Proof. Suppose that k of the d variables $x_1 = \cdots = x_k = 1$ and the rest are 0. Then, it follows

$$\sigma_1 = k \,, \quad \text{and} \quad \sigma_2 = \frac{\sigma_1^2 - \sigma_1}{2} = \frac{k(k-1)}{2} \,.$$

Even degree *

Proof cont'd. Let $l = \lfloor \frac{k}{2} \rfloor$, $m_d = \lfloor \frac{d-2}{2} \rfloor$, $N = \min(l, m_d)$ and

$$A := \min_{w_1,\dots,w_{n_d} \in \mathbb{B}} \sum_{i=1}^{m_d} w_i(-2\sigma_1 + 4i - 1) + \sigma_2 = \sum_{i=1}^{m_d} \min(0, -2k + 4i - 1) + \sigma_2 \,.$$

- If k is even ($k = 2l$): $-2k + 4i - 1 < 0 \;\Leftrightarrow\; 4i < 4l + 1 \;\Leftrightarrow\; i \leqslant l$.
- If k is odd ($k = 2l + 1$): $-2k + 4i - 1 < 0 \;\Leftrightarrow\; 4i < 4l + 3 \;\Leftrightarrow\; i \leqslant l$.

Thus,

$$A = \sum_{i=1}^{N} (-2k + 4i - 1) + \frac{k(k-1)}{2} = 2N^2 - N(2k - 1) + \frac{k(k-1)}{2} \,.$$

Note that if $k \leqslant d - 2$, then

$$l = \left\lfloor \frac{k}{2} \right\rfloor \leqslant \left\lfloor \frac{d-2}{2} \right\rfloor = m_d \,, \text{ hence } N = l \,.$$

Even degree *

Proof cont'd. We know that $A = 2N^2 - N(2k - 1) + k(k-1)/2$.

- $k \leqslant d - 2$.

$$\begin{aligned} k &= 2l : & A &= 2l^2 - l(2 \cdot 2l - 1) + \frac{2l(2l-1)}{2} = 0 \,. \\ k &= 2l + 1 : & A &= 2l^2 - l(2(2l+1) - 1) + \frac{(2l+1)(2l+1-1)}{2} = 0 \,. \end{aligned}$$

- $k = d - 1$. Thus, $l = \lfloor \frac{k}{2} \rfloor = \lfloor \frac{d-1}{2} \rfloor = \lfloor \frac{d-2}{2} \rfloor = m_d$, and $N = l = m_d$.

$$\begin{aligned} A &= 2m_d^2 - m_d(2(d-1) - 1) + \frac{(d-1)(d-2)}{2} \\ &= 2m_d^2 - m_d(2(2m_d + 1) - 1) + (2m_d + 1)m_d = 0 \,. \end{aligned}$$

- $k = d = 2l$: $l = \lfloor \frac{d}{2} \rfloor = \lfloor \frac{d-2}{2} \rfloor + 1 = m_d + 1$. Thus, $N = m_d$.

$$A = 2(l-1)^2 - (l-1)(4l-1) + l(2l-1) = 2l^2 - 4l + 2 - 4l^2 + 5l - 1 + 2l^2 - l = 1 \,.$$

Therefore $x_1 \cdots x_d = A$.

Even degree *

Proof cont'd. Since d is even, $n_d = \lfloor \frac{d-1}{2} \rfloor = \lfloor \frac{d-2}{2} \rfloor = m_d$.

$$\begin{aligned} a x_1 \cdots x_d &= aA \\ &= a \min_{w_1,\dots,w_{n_d} \in \mathbb{B}} \sum_{i=1}^{m_d} w_i(-2\sigma_1 + 4i - 1) + a\sigma_2 \\ &= \min_{w_1,\dots,w_{n_d} \in \mathbb{B}} a \left(\sum_{i=1}^{n_d} w_i(-2\sigma_1 + 4i - 1) \right) + a\sigma_2 \,, \end{aligned}$$

which completes the proof in the even-degree case. □

Odd degree *

Theorem 2. *Let $x_1, \dots, x_d \in \mathbb{B}$, where d is odd. For $a > 0$*

$$a x_1 \cdots x_d = \min_{w_1,\dots,w_{n_d} \in \mathbb{B}} a \left(\sum_{i=1}^{n_d - 1} w_i(-2\sigma_1 + 4i - 1) + w_{n_d}(-\sigma_1 + 2n_d - 1) \right) + a\sigma_2,$$

where n_d, σ_1 and σ_2 are defined as before.

Proof. We will use the notations defined in the previous proof. When d is odd, then $m_d = \lfloor \frac{d-2}{2} \rfloor = \lfloor \frac{d-1}{2} \rfloor - 1 = n_d - 1$. Therefore,

$$\begin{aligned} &\min_{w_1,\dots,w_{n_d} \in \mathbb{B}} \left(\sum_{i=1}^{n_d - 1} w_i(-2\sigma_1 + 4i - 1) + w_{n_d}(-\sigma_1 + 2n_d - 1) \right) + \sigma_2 \\ =& \min_{w_1,\dots,w_{n_d - 1} \in \mathbb{B}} \left(\sum_{i=1}^{m_d} w_i(-2\sigma_1 + 4i - 1) \right) + \sigma_2 + \min_{w_{n_d} \in \mathbb{B}} (-\sigma_1 + 2n_d - 1) \\ =& A + \min(0, -\sigma_1 + 2n_d - 1) \,. \end{aligned}$$

Odd degree *

Proof cont'd. Again, suppose that k of the d variables $x_1 = \cdots = x_k = 1$ and the rest are 0. Since $d = 2\lfloor \frac{d-1}{2} \rfloor + 1 = 2n_d + 1$, it follows that

$$-\sigma_1 + 2n_d - 1 = -k + d - 2 \geqslant 0 \quad \Leftrightarrow \quad k \leqslant d - 2 \,.$$

- $k \leqslant d - 2$. Thus, $A = 0$ and $-\sigma_1 + 2n_d - 1 \geqslant 0$. Therefore,

$$x_1 \cdots x_d = 0 \,, \quad \text{and} \quad A + \min(0, -\sigma_1 + 2n_d - 1) = 0 + 0 = 0 \,.$$

- $k = d - 1$. Thus $-\sigma_1 + 2n_d - 1 = -k + 2\lfloor \frac{d-1}{2} \rfloor - 1 = -k + k - 1 = -1$. Also, $m_d = n_d - 1 = \lfloor \frac{d-1}{2} \rfloor - 1 = \lfloor \frac{k}{2} \rfloor - 1 = l - 1$. Moreover, $N = \min(l, m_d) = l - 1$ and $k = 2l$, it follows that

$$A = 2N^2 - N(2k - 1) + \frac{k(k-1)}{2} = 2(l-1)^2 - (l-1)(4l-1) + l(2l-1) = 1 \,.$$

Therefore, $x_1 \cdots x_d = 0$, and $A + \min(0, -\sigma_1 + 2n_d - 1) = 1 - 1 = 0$.

Odd degree *

Proof cont'd.

- $k = d$. Thus $-\sigma_1 + 2n_d - 1 = -k + 2\lfloor \frac{d-1}{2} \rfloor - 1 = -k + k - 1 - 1 = -2$, and $N = m_d = l - 1$, $k = 2l + 1$, thus

$$A = 2N^2 - N(2k - 1) + \frac{k(k-1)}{2} = 2(l-1)^2 - (l-1)(4l+1) + (2l+1)l = 3; .$$

Therefore $x_1 \cdots x_d = 1$ and $A + \min(0, -\sigma_1 + 2n_d - 1) = 3 - 2 = 1$.

□

As we have seen (in the reduction by minimization),

$$axyz = \min_{w \in \mathbb{B}} a\big(w(x + y + z - 1) - (x + y + z) + (xy + yz + zx) + 1\big) \,.$$

However, by applying the theorem, we obtain

$$= \min_{w_1 \in \mathbb{B}} a w_1\big(-(x + y + z - 1) + 1\big) + a(xy + yz + zx) \,.$$

General case

Consider a monomial $ax_1 \cdots x_d$ of degree d. We define the elementary symmetric polynomials in these variables as

$$\sigma_1 = \sum_{i=1}^{d} x_i \,, \quad \text{and} \quad \sigma_2 = \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} x_i x_j = \frac{\sigma_1(\sigma_1 - 1)}{2} \,.$$

In summary,

- If $a < 0$: $ax_1 \cdots x_d = \min_{w \in \mathbb{B}} aw(\sigma_1 - (d - 1))$.
- If $a > 0$:

$$ax_1 \cdots x_d = a \min_{w_1,\dots,w_{n_d} \in \mathbb{B}} \sum_{i=1}^{n_d} w_i(c_{i,d}(-\sigma_1 + 2i) - 1) + a\sigma_2 \,,$$

where $$n_d = \left\lfloor \frac{d-1}{2} \right\rfloor \,, \quad c_{i,d} = \begin{cases} 1, & \text{if } d \text{ is odd and } i = n_d, \\ 2, & \text{otherwise.} \end{cases}$$

General case

- Any pseudo-Boolean function can be uniquely written as a polynomial in binary variables.
- Each monomial can be reduced to a quadratic polynomial by making use of the above technique.

Therefore, the whole function can be reduced to a quadratic polynomial, such that if any assignment of values to the variables in the reduced polynomial achieves its minimum, the assignment restricted to the original variables achieves a minimum of the original function.

Note that the reduction is valid either the function is submodular or not.

The number of additional variables (per clique) in the worst case is exponential in d. For instance, with a clique of size five, there can be up to one quintic, five quartic, and ten cubic terms, and 17 new variables could be needed.

Transforming multi-label functions

Transforming multi-label functions

Let $\mathcal{V}$ and $\mathcal{L}$ be the set of pixels and labels, respectively. Consider the following energy function on a labeling $\mathbf{y} \in \mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_{|\mathcal{V}|} = \mathcal{L}^{\mathcal{V}}$

$$E(\mathbf{y}) = \sum_{c \in \mathcal{C}} E_c(y_c) ,$$

where $\mathcal{C}$ is the set of cliques and $E_c(y_c)$ denotes the local energy depending on the labels $y_c \in \mathcal{L}^c$.

The goal: is to optimize higher-order energies with more than two labels.
One can apply *fusion move* (see Lecture 13.), where in each iteration the current labeling and a proposed one is fused by minimizing a pseudo-Boolean energy.

For a **proposed labeling** $\mathbf{p} \in \mathcal{Y}$, we consider a binary labeling $\mathbf{z} \in \mathbb{B}^{\mathcal{V}}$ such that for all $v \in \mathcal{V}$

$$y'_v = \begin{cases} y_v, & \text{if } z_v = 0 \\ p_v, & \text{if } z_v = 1 . \end{cases}$$

Transforming multi-label functions

Therefore,

$$y'_v = (1 - z_v) y_v + z_v p_v .$$

We define a pseudo-Boolean function

$$f(\mathbf{z}) = \sum_{c \in \mathcal{C}} E_c\big(\lambda_c(z_c; y_c, p_c)\big) ,$$

where for all $c \in \mathcal{C}$,

$$\big(\lambda_c(z_c; y_c, p_c)\big)_v = y'_v \qquad \forall v \in c .$$

1. The polynomial $f(\mathbf{z})$ is reduced into a quadratic one.
2. The QPBO method (see Lecture 8.) can be used to minimize $f(\mathbf{z})$, and it results an assignment of $0, 1$, or -1 to each pixel v.
3. y_v is updated to p_v if 1 is assigned to v, otherwise it remains unchanged.
4. Iterate the process until some convergence criterion is met.

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