Combinatorial Optimization in Computer Vision (IN2245)

Frank R. Schmidt
Csaba Domokos

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18. FastPD: Approximate Labeling via Primal-Dual Schema

Multi-label problem

Multi-label problem revisited

Consider an *undirected graphical model* given by $G = (\mathcal{V}, \mathcal{E})$ which takes values from an **arbitrary** (finite) label set $\mathcal{L}$.

More specially, assume that the corresponding *energy function* is given by

$$E(\mathbf{x}) = \sum_{i \in \mathcal{V}} \varphi_i(\mathbf{x}_i) + \sum_{(i,j) \in \mathcal{E}} w_{ij} \cdot d(\mathbf{x}_i, \mathbf{x}_j) ,$$

where φ_i stands for the *data term*, $w_{ij} \in \mathbb{R}$ are *weighting factors*, and d is a *metric* or a *semi-metric* (i.e. the triangle inequality is not necessary satisfied).

We have already seen some applications in Computer Vision corresponding to this energy function (e.g., stereo matching, image denoising, optical flow).

As we have discussed (in Lecture 13) one possible way to *approximately* solve this problem is to apply *move making algorithms* (e.g., α-expansion).

Equivalent integer linear program

We are generally interested to find a *MAP labelling* $\mathbf{x}^*$:

$$\mathbf{x}^* \in \operatorname*{argmin}_{\mathbf{x} \in \mathcal{L}^{|\mathcal{V}|}} E(\mathbf{x}) = \operatorname*{argmin}_{\mathbf{x} \in \mathcal{L}^{|\mathcal{V}|}} \left\{ \sum_{i \in \mathcal{V}} \varphi_i(x_i) + \sum_{(i,j) \in \mathcal{E}} w_{ij} \cdot d(x_i, x_j) \right\} .$$

This can be equivalently written as an **integer linear program** (ILP):

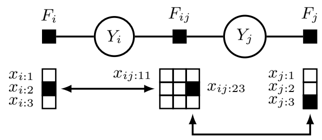
$$\min_{x_{i:\alpha}, x_{ij:\alpha\beta}} \sum_{i \in \mathcal{V}} \sum_{\alpha \in \mathcal{L}} \varphi_i(a) x_{i:\alpha} + \sum_{(i,j) \in \mathcal{E}} w_{ij} \sum_{\alpha,\beta \in \mathcal{L}} d(\alpha,\beta) x_{ij:\alpha\beta}$$

$$\begin{aligned} \text{subject to} \quad & \textstyle\sum_{\alpha \in \mathcal{L}} x_{i:\alpha} &= 1 \quad & \forall i \in \mathcal{V} \\ & \textstyle\sum_{\alpha \in \mathcal{L}} x_{ij:\alpha\beta} &= x_{j:\beta} \quad & \forall \beta \in \mathcal{L}, (i,j) \in \mathcal{E} \\ & \textstyle\sum_{\beta \in \mathcal{L}} x_{ij:\alpha\beta} &= x_{i:\alpha} \quad & \forall \alpha \in \mathcal{L}, (i,j) \in \mathcal{E} \\ & x_{i:\alpha}, x_{ij:\alpha\beta} \in \mathbb{B} & & \forall \alpha, \beta \in \mathcal{L}, (i,j) \in \mathcal{E} \end{aligned}$$

$x_{i:\alpha}$ indicates whether vertex i is assigned label α, while $x_{ij:\alpha\beta}$ indicates whether (neighboring) vertices i, j are assigned labels α, β, respectively.

Interpretation of the constraints

Let us assume that $\mathcal{L} = \{1, 2, 3\}$ and consider the following example:



Uniqueness: The constraint $\sum_{\alpha \in \mathcal{L}} x_{i:\alpha} = 1$ simply express the fact that each vertex must receive exactly one label.

Consistency: The constraints $\sum_{\alpha \in \mathcal{L}} x_{ij:\alpha\beta} = x_{j:\beta}$ and $\sum_{\beta \in \mathcal{L}} x_{ij:\alpha\beta} = x_{i:\alpha}$ maintain consistency between variables, i.e. if $x_{i:\alpha} = 1$ and $x_{j:\beta} = 1$ holds true, then these constraints force $x_{ij:\alpha\beta} = 1$ to hold true as well.

FastPD algorithm vs. α-expansion

The *FastPD algorithm* is a max-flow based combinatorial method which is suitable for *approximate optimization* of a very wide class of MRFs.

It utilizes tools from the **duality theory of linear programming** in order to provide a **more general view of move making techniques**.

This algorithm solves similar problems as the α-expansion (which is included merely as a special case), but it has some advantages:

- **It is more general**: It can be applied for a much wider class of problems, e.g., MRFs with non-metric potentials.
- **It is more efficient**: It is guaranteed that the generated solution will always be within a known factor of the global optimum. In practice, these bounds prove to be very tight (i.e. very close to 1).
- **It is conceptually more elegant**.

Primal-dual LP

Primal-dual LP revisited

Consider a linear program (given in **standard form**):

$$\min_{\mathbf{x}\in\mathbb{R}^n}\langle\mathbf{c},\mathbf{x}\rangle$$
$$\text{subject to } \mathbf{A}\mathbf{x}=\mathbf{b}, \mathbf{x}\geqslant\mathbf{0}\,,$$

for a *constraint matrix* $\mathbf{A}\in\mathbb{R}^{m\times n}$, a *constraint vector* $\mathbf{b}\in\mathbb{R}^m$ and a *cost vector* $\mathbf{c}\in\mathbb{R}^n$.

The *dual LP* is defined as

$$\max_{\mathbf{y}\in\mathbb{R}^m}\langle\mathbf{b},\mathbf{y}\rangle$$
$$\text{subject to } \mathbf{A}^T\mathbf{y}\leqslant\mathbf{c}\,.$$

Due to weak duality $\langle\mathbf{b},\mathbf{y}\rangle\leqslant\langle\mathbf{c},\mathbf{x}\rangle$ is held for feasible solutions.

For more details you may refer to Lecture 7.

LP relaxation

The ILP defined before is in general NP-hard. Therefore we deal with the **LP relaxation** of our ILP. The relaxed LP can be written in *standard form* as follows:

$$\min_{x_{i:\alpha},x_{ij:\alpha\beta}}\langle\mathbf{c},\mathbf{x}\rangle$$
$$\text{subject to } \mathbf{A}\mathbf{x}=\mathbf{b}, \mathbf{x}\geqslant\mathbf{0}\,.$$

Reminder: The **lexicographical order relation $\prec$ on $\mathbb{N}^k$** is defined as

$$(u_1,\dots,u_k)\prec(v_1,\dots,v_k)\quad\Leftrightarrow\quad\exists l:\ \forall i<l\ (u_i=v_i) \text{ and } (u_l<v_l)\,.$$

Reminder: Assume $\mathbf{A}\in\mathbb{R}^{k\times l}$ and $\mathbf{B}\in\mathbb{R}^{m\times n}$, then the **Kronecker product $\mathbf{A}\otimes\mathbf{B}$** is the $km\times ln$ block matrix: $\mathbf{A}\otimes\mathbf{B}=\begin{bmatrix}a_{11}\mathbf{B} & \cdots & a_{1l}\mathbf{B}\\ \vdots & \ddots & \vdots\\ a_{k1}\mathbf{B} & \cdots & a_{kl}\mathbf{B}\end{bmatrix}$.

LP relaxation: cost function

$$\min_{x_{i:\alpha},x_{ij:\alpha\beta}}\langle\mathbf{c},\mathbf{x}\rangle \qquad \text{subject to } \mathbf{A}\mathbf{x}=\mathbf{b}, \mathbf{x}\geqslant\mathbf{0}\,.$$

We may write $\mathbf{x}=\begin{bmatrix}\mathbf{x}_1^T & \mathbf{x}_2^T\end{bmatrix}^T$, where

$$\mathbf{x}_1=\begin{bmatrix}x_{1:1} & \cdots & x_{1:m} & x_{2:1} & \cdots & x_{2:m} & x_{n:1} & \cdots & x_{n:m}\end{bmatrix}^T\in\mathbb{R}^{mn}\,,$$

where $n=|\mathcal{V}|$ and $m=|\mathcal{L}|$, and $\mathbf{x}_2\in\mathbb{R}^{|\mathcal{E}|m^2}$ is the vector consisting of all the variables $x_{ij:\alpha\beta}$ in *lexicographic order* based on the corresponding 4-tuples (i,j,α,β).

Similarly, we can write $\mathbf{c}=\begin{bmatrix}\mathbf{c}_1^T & \mathbf{c}_2^T\end{bmatrix}^T$, where

$$\mathbf{c}_1=\begin{bmatrix}\varphi_1(1) & \cdots & \varphi_1(m) & \cdots & \varphi_n(1) & \cdots & \varphi_n(m)\end{bmatrix}^T\in\mathbb{R}^{mn}\,,$$

and $\mathbf{c}_2\in\mathbb{R}^{|\mathcal{E}|m^2}$ is the vector consisting of the values $w_{ij}d(\alpha,\beta)$ in *lexicographic order* based on the corresponding 4-tuples (i,j,α,β).

Therefore, $\langle\mathbf{c},\mathbf{x}\rangle=\langle\mathbf{c}_1,\mathbf{x}_1\rangle+\langle\mathbf{c}_2,\mathbf{x}_2\rangle$.

LP relaxation: constraints

$$\min_{x_{i:\alpha},x_{ij:\alpha\beta}}\langle\mathbf{c},\mathbf{x}\rangle \qquad \text{subject to } \mathbf{A}\mathbf{x}=\mathbf{b}, \mathbf{x}\geqslant\mathbf{0}\,.$$

We can write the (uniqueness) constraints $\sum_{\alpha\in\mathcal{L}}x_{i:\alpha}=1$ for all $p\in\mathcal{V}$ as

$$\begin{bmatrix}\mathbf{I}_{n\times n}\otimes\mathbf{1}_m^T\end{bmatrix}\mathbf{x}_1=\mathbf{1}_n=:\mathbf{b}_1\,,$$

where $\mathbf{1}_n\in\mathbb{R}^n$ is the vector of all-ones.

We introduce the notation $\pi_{\mathcal{E}}(i,j)$ for the **index of an element** $(i,j)\in\mathcal{E}$ according to the lexicographic order $\prec$ on $\mathcal{E}$, that is

$$\pi_{\mathcal{E}}(i,j)\triangleq|\{(k,l)\in\mathcal{E}\mid(k,l)\prec(i,j)\}|\,.$$

LP relaxation: constraints

$$\min_{x_{i:\alpha},x_{ij:\alpha\beta}}\langle\mathbf{c},\mathbf{x}\rangle \qquad \text{subject to } \mathbf{A}\mathbf{x}=\mathbf{b}, \mathbf{x}\geqslant\mathbf{0}\,.$$

The (consistency) constraint $\sum_{\alpha\in\mathcal{L}}x_{ij:\alpha\beta}=x_{j:\beta}\ \Leftrightarrow\ -x_{j:\beta}+\sum_{\alpha\in\mathcal{L}}x_{ij:\alpha\beta}=0$ can be expressed as

$$\begin{bmatrix}-\mathbf{u}^T_{(j-1)m+\beta} & \sum_{\alpha\in\mathcal{L}}\mathbf{v}^T_{m^2\pi_{\mathcal{E}}(i,j)+(\alpha-1)m+\beta}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}=0\,,$$

where $\mathbf{u}_k\in\mathbb{R}^{mn}$ and $\mathbf{v}_k\in\mathbb{R}^{|\mathcal{E}|m^2}$ are **k^{th} standard unit vectors** whose k^{th} component is equal to one and all the other elements are equal to zero.

One can collect **all** the *consisteny constraints* as follows

$$\begin{bmatrix}-\mathbf{U} \mid \mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}=\mathbf{0}_{2|\mathcal{E}|m}=:\mathbf{b}_2\,,$$

where $\mathbf{U}\in\mathbb{R}^{2|\mathcal{E}|m\times mn}$ and $\mathbf{V}\in\mathbb{R}^{2|\mathcal{E}|m\times|\mathcal{E}|m^2}$.

LP relaxation: constraints

$$\min_{x_{i:\alpha},x_{ij:\alpha\beta}}\langle\mathbf{c},\mathbf{x}\rangle \qquad \text{subject to } \mathbf{A}\mathbf{x}=\mathbf{b}, \mathbf{x}\geqslant\mathbf{0}\,.$$

We can write all the constraints in a matrix-vector notation as follows.

$$\mathbf{A}\mathbf{x}=\left[\begin{array}{c|c}\mathbf{I}_{n\times n}\otimes\mathbf{1}_m^T & \mathbf{0}_{n\times|\mathcal{E}|m^2}\\ -\mathbf{U} & \mathbf{V}\end{array}\right]\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}=\begin{bmatrix}\mathbf{1}_n\\ \mathbf{0}_{2|\mathcal{E}|m}\end{bmatrix}=\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{bmatrix}=\mathbf{b}\,.$$

Hence, $\mathbf{A}\in\mathbb{R}^{n+2|\mathcal{E}|m\times mn+|\mathcal{E}|m^2}$ is a **sparse matrix** with elements -1,0 and 1, furthermore $\mathbf{b}\in\mathbb{R}^{n+2|\mathcal{E}|m}$, where the first mn elements are equal to one and the others are equal to zero.

Column consistency: We assume that the first $|\mathcal{E}|m$ rows of $\mathbf{U}$ and $\mathbf{V}$ correspond to the constraints $\sum_{\alpha\in\mathcal{L}}x_{ij:\alpha\beta}=x_{j:\beta}$ enumerated in *lexicographic order* based on (i,j,β).

Row consistency: the second half of the rows in $\mathbf{U}$ and $\mathbf{V}$ correspond to the constraints $\sum_{\beta\in\mathcal{L}}x_{ij:\alpha\beta}=x_{i:\alpha}$ enumerated in lexicographic based on (i,j,α).

Dual LP

$$\max_{y_i,y_{ij:\alpha},y_{ji:\beta}}\langle\mathbf{b},\mathbf{y}\rangle \qquad \text{subject to } \mathbf{A}^T\mathbf{y}\leqslant\mathbf{c}\,.$$

Note that the dual variables y_i for all $i\in\mathcal{V}$ and $y_{ij:\alpha}$, $y_{ji:\beta}$ for all $(i,j)\in\mathcal{E}$, $\alpha,\beta\in\mathcal{L}$ correspond to the constraints of the primal LP.

We can write $\mathbf{y}=\begin{bmatrix}\mathbf{y}_1^T & \mathbf{y}_2^T & \mathbf{y}_3^T\end{bmatrix}^T$, where $\mathbf{y}_1=\begin{bmatrix}y_1 & \cdots & y_n\end{bmatrix}^T\in\mathbb{R}^n$, and $\mathbf{y}_2\in\mathbb{R}^{|\mathcal{E}|m}$ and $\mathbf{y}_3\in\mathbb{R}^{|\mathcal{E}|m}$ are the vectors consisting of the variables $y_{ji:\beta}$ and $y_{ij:\alpha}$ in the same order as it is defined in the case of the primal LP.

The cost function results in

$$\langle\mathbf{b},\mathbf{y}\rangle=\langle\mathbf{b}_1,\mathbf{y}_1\rangle+\langle\mathbf{b}_2,\begin{bmatrix}\mathbf{y}_2^T & \mathbf{y}_3^T\end{bmatrix}^T\rangle=\langle\mathbf{1}_n,\mathbf{y}_1\rangle=\sum_{i=1}^n y_i\,.$$

The constraints $\mathbf{A}^T\mathbf{y}\leqslant\mathbf{c}$ are given by

$$\mathbf{A}^T\mathbf{y}=\left[\begin{array}{c|c}\mathbf{I}_{n\times n}\otimes\mathbf{1}_m & -\mathbf{U}^T\\ \mathbf{0}_{|\mathcal{E}|m^2\times n} & \mathbf{V}^T\end{array}\right]\mathbf{y}\leqslant\begin{bmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{bmatrix}=\mathbf{c}\,.$$

Dual LP

$$\max_{y_i,y_{ij:\alpha},y_{ji:\beta}}\langle\mathbf{1}_n,\mathbf{y}_1\rangle$$

$$\text{subject to } \left[\begin{array}{c|c}\mathbf{I}_{n\times n}\otimes\mathbf{1}_m & -\mathbf{U}^T\\ \mathbf{0}_{|\mathcal{E}|m^2\times n} & \mathbf{V}^T\end{array}\right]\mathbf{y}\leqslant\begin{bmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{bmatrix}.$$

Or equivalently, we can formulate the dual LP as

$$\max_{y_i,y_{ij:\alpha},y_{ji:\beta}}\sum_{i\in\mathcal{V}}y_i$$

$$\begin{aligned}\text{subject to}\quad & y_i-\sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}}y_{ij:\alpha} \leqslant\varphi_i(\alpha) && \forall i\in\mathcal{V},\alpha\in\mathcal{L}\\ & y_{ij:\alpha}+y_{ji:\beta} \leqslant w_{ij}d(\alpha,\beta) && \forall(i,j)\in\mathcal{E},\alpha,\beta\in\mathcal{L}\end{aligned}$$

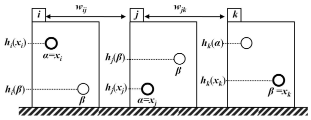
An intuitive view of the dual variables

We will use the notation $x_i \in \mathcal{L}$ for the **active label** given the vertex $i \in \mathcal{V}$.

For each vertex we have a different copy of all labels in $\mathcal{L}$. It is assumed that all these labels represent **balls** floating at certain heights relative to a *reference plane*.

For this sake we introduce **height variables** defined as

$$h_i(\alpha) = \varphi_i(\alpha) + \sum_{j \in \mathcal{V}, (i,j) \in \mathcal{E}} y_{ij:\alpha} \,.$$



The constraints $y_i - \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y_{ij:\alpha} \leqslant \varphi_i(\alpha)$ can be equivalently written as

$$y_i \leqslant \varphi_i(\alpha) + \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y_{ij:\alpha} = h_i(\alpha) \qquad \forall i \in \mathcal{V}, \alpha \in \mathcal{L} \,.$$

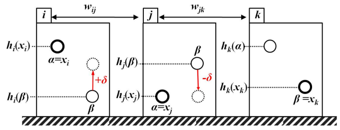
Since our objective is to maximize $\sum_{i \in \mathcal{V}} y_i$, the following relation holds

$$y_i = \min_{\alpha \in \mathcal{L}} h_i(\alpha) \qquad \forall i \in \mathcal{V} \,.$$

Balance variables and load

We will refer to the variables $y_{ij:\alpha}$, $y_{ji:\beta}$ as **balance variables**. Specially, the pair of $y_{ij:\alpha}$, $y_{ji:\alpha}$ is called **conjugate balance variables**.

The *balls* are not static, but may move in pairs through updating pairs of *conjugate balance variables* as $h_i(\alpha) = \varphi_i(\alpha) + \sum_{j \in \mathcal{V}, (i,j) \in \mathcal{E}} y_{ij:\alpha}$. Therefore, the role of *balance variables* is to raise or lower labels.



It is due to $y_{ij:\alpha} + y_{ji:\alpha} \leqslant w_{ij} d(\alpha, \alpha) = 0 \quad \Rightarrow \quad y_{ij:\alpha} \leqslant -y_{ji:\alpha}$.

We will call the variables $y_{ij:x_i}$ as **active balance variable** and use the following notation for the **"load"** between neighbors i, j, defined as

$$\text{load}_{ij} = y_{ij:x_i} + y_{ji:x_j} \,.$$

Primal-dual LP for multi-label problem

The (relaxed) primal LP:

$$\min_{x_{i:\alpha}, x_{ij:\alpha\beta} \geqslant 0} \sum_{i \in \mathcal{V}} \sum_{\alpha \in \mathcal{L}} \varphi_i(\alpha) x_{i:\alpha} + \sum_{(i,j) \in \mathcal{E}} w_{ij} \sum_{\alpha, \beta \in \mathcal{L}} d(\alpha, \beta) x_{ij:\alpha\beta}$$

$$\begin{aligned} \text{subject to} \quad & \textstyle\sum_{\alpha \in \mathcal{L}} x_{i:\alpha} && = 1 && \forall i \in \mathcal{V} \\ & \textstyle\sum_{\alpha \in \mathcal{L}} x_{ij:\alpha\beta} && = x_{j:\beta} && \forall \beta \in \mathcal{L}, (i,j) \in \mathcal{E} \\ & \textstyle\sum_{\beta \in \mathcal{L}} x_{ij:\alpha\beta} && = x_{i:\alpha} && \forall \alpha \in \mathcal{L}, (i,j) \in \mathcal{E} \end{aligned}$$

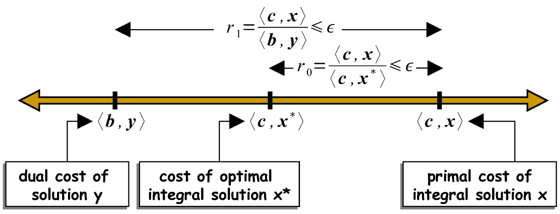
The dual LP:

$$\max_{y_i, y_{ij:\alpha}, y_{ji:\beta}} \sum_{i \in \mathcal{V}} y_i$$

$$\begin{aligned} \text{subject to} \quad & y_i - \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y_{ij:\alpha} && \leqslant \varphi_i(\alpha) && \forall i \in \mathcal{V}, \alpha \in \mathcal{L} \\ & y_{ij:\alpha} + y_{ji:\beta} && \leqslant w_{ij} d(\alpha, \beta) && \forall (i,j) \in \mathcal{E}, \alpha, \beta \in \mathcal{L} \end{aligned}$$

Primal-dual principle

Primal-dual principle



Theorem 1. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are integral-primal and dual feasible solutions satisfying:*

$$\langle \mathbf{c}, \mathbf{x} \rangle \leqslant \epsilon \langle \mathbf{b}, \mathbf{y} \rangle$$

for $\epsilon \geqslant 1$, *then* $\mathbf{x}$ *is an* **ϵ-approximation** *to the optimal integral solution* $\mathbf{x}^*$, *that is*

$$\langle \mathbf{c}, \mathbf{x}^* \rangle \leqslant \langle \mathbf{c}, \mathbf{x} \rangle \leqslant \epsilon \langle \mathbf{b}, \mathbf{y} \rangle \leqslant \epsilon \langle \mathbf{c}, \mathbf{x}^* \rangle \,.$$

The relaxed complementary slackness

One way to estimate a pair $(\mathbf{x}, \mathbf{y})$ satisfying the fundamental inequality $\langle \mathbf{c}, \mathbf{x} \rangle \leqslant \epsilon \langle \mathbf{b}, \mathbf{y} \rangle$ relies the **complementary slackness principle**.

Theorem 2. *If the pair* $(\mathbf{x}, \mathbf{y})$ *of integral-primal and dual feasible solutions satisfies the so-called* **relaxed primal complementary slackness conditions***:*

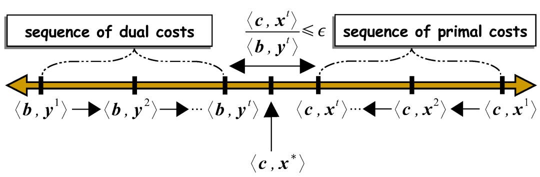
$$\forall j: \ (x_j > 0) \quad \Rightarrow \quad \left(\sum_i a_{ij} y_i \geqslant \frac{c_j}{\epsilon_j} \right) \,,$$

then $(\mathbf{x}, \mathbf{y})$ *also satisfies* $\langle \mathbf{c}, \mathbf{x} \rangle \leqslant \epsilon \langle \mathbf{b}, \mathbf{y} \rangle$ *with* $\epsilon = \max_j \epsilon_j$ *and therefore* $\mathbf{x}$ *is an* ϵ*-approximation to the optimal integral solution* $\mathbf{x}^*$.

Proof. Exercise. □

Primal-dual schema

Primal-dual schema



Typically, primal-dual ϵ-approximation algorithms construct a sequence $(\mathbf{x}^k, \mathbf{y}^k)_{k=1,\dots,t}$ of primal and dual solutions until the elements $\mathbf{x}^t$, $\mathbf{y}^t$ of the last pair are both **feasible** and **satisfy the relaxed primal complementary slackness conditions**, hence the condition $\langle \mathbf{c}, \mathbf{x} \rangle \leqslant \epsilon \langle \mathbf{b}, \mathbf{y} \rangle$ will be also fulfilled.

Pseudo-code of the FastPD algorithm


```
1:  [x, y] ←Init_Primals_Duals()
2:  labelChange ←false
3:  for all α ∈ ℒ do                                ▷ α-iteration
4:      y ←PreEdit_Duals(α, x, y)
5:      [x′, y′] ←Update_Duals_Primals(α, x, y)
6:      y′ ←PostEdit_Duals(α, x′, y′)
7:      if x′ ≠ x then
8:          labelChange ←true
9:      end if
10:     x ← x′; y ← y′
11: end for
12: if labelChange then
13:     goto 2
14: end if
15: y^fit ←Dual_Fit(y)
```


PD1

Complementary slackness conditions

From now on, in case of Algorithm PD1, we only assume that $d(\alpha,\beta)=0 \Leftrightarrow \alpha=\beta$, and $d(\alpha,\beta)\geqslant 0$.

The *complementary slackness conditions* reduces to

$$y_i - \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y_{ij:x_i} \geqslant \frac{\varphi_i(x_i)}{\epsilon_1} \quad\Rightarrow\quad y_i \geqslant \frac{\varphi_i(x_i)}{\epsilon_1} + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y_{ij:x_i}$$

$$y_{ij:x_i} + y_{ji:x_j} \geqslant \frac{w_{ij}d(x_i,x_j)}{\epsilon_2}$$

for specific values of $\epsilon_1, \epsilon_2 \geqslant 1$.

If $x_i = x_j = \alpha$ for neighboring i, j, then

$$0 = w_{ij:\alpha}d(\alpha,\alpha) \geqslant y_{ij:i\alpha} + y_{ij:j\alpha} \geqslant \frac{w_{ij}d(\alpha,\alpha)}{\epsilon_2} = 0\ ,$$

therefore we get that $y_{ij:\alpha} = -y_{ij:\alpha}$.

Complementary slackness conditions

We have known that $y_i = \min_{\alpha\in\mathcal{L}} h_i(\alpha)$. If $\epsilon_1 = 1$, then we get

$$y_i \geqslant \varphi_i(x_i) + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y_{ij:x_i} = h_i(x_i)\ .$$

Therefore

$$h_i(x_i) = \min_{\alpha\in\mathcal{L}} h_i(\alpha)\ , \qquad (1)$$

which means that, at each vertex, **the active label should have the lowest height**.

If $\epsilon_2 = \epsilon_{\text{app}} := \frac{2d_{\max}}{d_{\min}}$, then the *complementary condition* simply reduces to:

$$y_{ij:x_i} + y_{ij:x_j} \geqslant \frac{w_{ij}d(x_i,x_j)}{\epsilon_{\text{app}}}\ . \qquad (2)$$

It requires that any **two active labels should be raised proportionally to their "load"**.

Feasibility constraints

To ensure feasibility of $\mathbf{y}$, PD1 enforces for any $\alpha\in\mathcal{L}$:

$$y_{ij:\alpha} \leqslant w_{ij}d_{\min}/2 \quad \text{where} \quad d_{\min} = \min_{\alpha\neq\beta} d(\alpha,\beta) \qquad (3)$$

says that **there is an upper bound on how much we can raise a label**.

Hence, we get the feasibility condition

$$y_{ij:\alpha} + y_{ji:\beta} \leqslant 2w_{ij}d_{\min}/2 = w_{ij}d_{\min} \leqslant w_{ij}d(\alpha,\beta)\ .$$

Moreover the algorithm **keeps the active balance variables non-negative**, that is $y_{ij:x_i} \geqslant 0$ for all $i\in\mathcal{V}$.

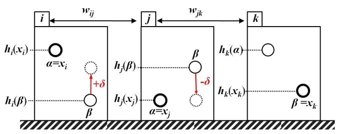
The *proportionality condition* (2) will be also fulfilled as $y_{ij:x_i}, y_{ij:x_j} \geqslant 0$ and if $y_{ij:x_i} = \frac{w_{ij}d_{\min}}{2}$, then

$$y_{ij:x_i} \geqslant \frac{w_{ij}d_{\min}}{2}\frac{d(x_i,x_j)}{d_{\max}} = \frac{w_{ij}d(x_i,x_j)}{\frac{2d_{\max}}{d_{\min}}} = \frac{w_{ij}d(x_i,x_j)}{\epsilon_{\text{app}}}\ .$$

Subroutine Init_Primals_Duals() *


```
function INIT_PRIMALS_DUALS
    x is simply initialized by a random label assignment        ▷ Init primals
    for all (i, j) ∈ ℰ with x_i ≠ x_j do                         ▷ Init duals
        y_ij:x_i ← w_ij d(x_i, x_j)/2
        y_ji:x_i ← −w_ij d(x_i, x_j)/2
        y_ji:x_j ← w_ij d(x_i, x_j)/2
        y_ij:x_j ← −w_ij d(x_i, x_j)/2
    end for
    for all i ∈ 𝒱 do
        y_i ← min_{α∈ℒ} h_i(α)
    end for
    return [x, y]
end function
```


Update primal and dual variables



Dual variables update: Given the current active labels, any non-active label is raised, until it either reaches the active label, or attains the maximum raise allowed by the upper bound (3).

Primal variables update: Given the new heights, there might still be vertices whose active labels are not at the lowest height. For each such vertex i, we select a non-active label, which is below x_i, but has already reached the maximum raise allowed by the upper bound (3).

The optimal update of the α-heights can be simulated by pushing the **maximum amount of flow** through a directed graph $G = (\mathcal{V}\cup\{s,t\}, \mathcal{E}', \mathcal{C})$.

Flow construction: n-edges

For each $(i,j)\in\mathcal{E}$, we insert two directed edges ij and ji into $\mathcal{E}'$.

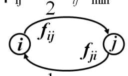
The flow value f_{ij}, f_{ij} represent respectively the **increase, decrease of balance variable** $y_{pq:\alpha}$:

$$y'_{ij:\alpha} = y_{ij:\alpha} + f_{ij} - f_{ji} \quad \text{and} \quad y'_{ji:\alpha} = -y'_{ij:\alpha}\ .$$

According to (3), the capacities cap_{ij} and cap_{ji} are set based on

$$\text{cap}_{ij} + y_{ij:\alpha} = \frac{1}{2}w_{ij}d_{\min} = \text{cap}_{ji} + y_{ji:\alpha}\ .$$

$$\text{cap}_{ij} = \frac{1}{2}w_{ij}d_{\min} - y_{ij:\alpha}$$



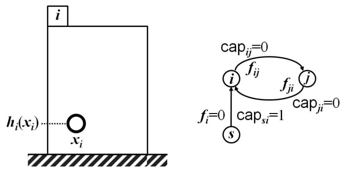
$$\text{cap}_{ji} = \frac{1}{2}w_{ij}d_{\min} - y_{ji:\alpha}$$

Flow construction: n-edges

If α is already the active label of i (or j), then label α at i (or j) need not move.

Therefore, $y'_{ij:\alpha} = y_{ij:\alpha}$ and $y'_{ji:\alpha} = y_{ji:\alpha}$, that is

$$x_i = \alpha \text{ or } x_j = \alpha \quad \Rightarrow \quad \mathsf{cap}_{ij} = \mathsf{cap}_{ji} = 0\ .$$



Flow construction: t-edges

Each node $i \in \mathcal{V}' - \{s,t\}$ connects to either the source node s or the sink node t (but not to both of them).
There are three possible cases to consider:

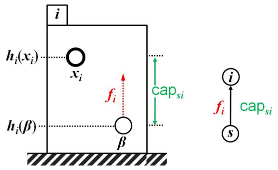
Case 1 ($h_i(\alpha) < h_i(x_i)$): we want to raise label α as much as it reaches label x_i.
We connect source node s to node i.
Due to the flow conservation property, $f_i = \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} (f_{ij} - f_{ji})$.
The flow f_i through that edge will then represent the total relative raise of label α:

$$\begin{aligned} h_i(\alpha) + f_i &= \Big(\varphi_i(\alpha) + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y_{ij:\alpha}\Big) + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} (f_{ij} - f_{ji}) \\ &= \Big(\varphi_i(\alpha) + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y_{ij:\alpha}\Big) + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} (y'_{ij:\alpha} - y_{ji:\alpha}) \\ &= \varphi_p(\alpha) + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y'_{ij:\alpha} = h'_i(\alpha)\ . \end{aligned}$$

Flow construction: t-edges

We need to raise α only as high as the current active label of i, but not higher than that, we therefore set:

$$\mathsf{cap}_{si} = h_i(x_i) - h_i(\alpha)\ .$$



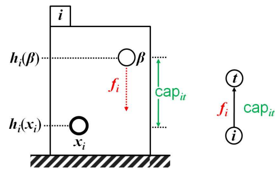
Flow construction: t-edges

Case 2 ($h_i(\alpha) \geqslant h_i(x_i)$ **and** $c \neq x_i$): we can then afford a decrease in the height of α at i, as long as α remains above x_p.

We connect i to the sink node t through directed edge it.

The flow f_i through edge it will equal the total relative decrease in the height of α:

$$\begin{aligned} h'_i(\alpha) &= h_i(\alpha) - f_i \\ \mathsf{cap}_{it} &= h_i(\alpha) - h_i(x_i)\ . \end{aligned}$$

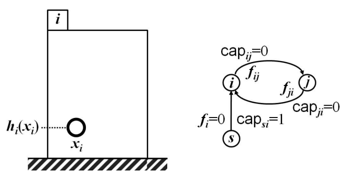


Flow construction: t-edges

Case 3 ($\alpha = x_i$): we want to keep the height of α fixed at the current iteration.

Note that the capacities of the *n-edges* for p are set to 0, since i has the *active label*. Therefore, $f_i = 0$ and $h'_{ij:\alpha} = h_{ij:\alpha}$.

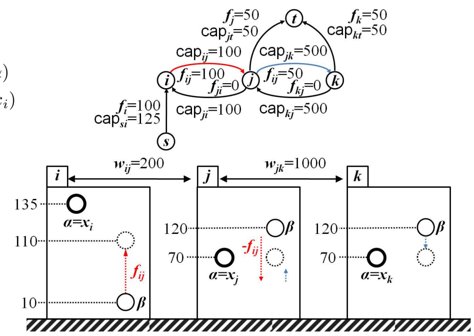
By convention $\mathsf{cap}_{ij} := 1$.



Reassign rule

Label α will be the new label of i (i.e. $x'_i = \alpha$) iff there exists unsaturated path between the source node s and node i. In all other cases, i keeps its current label (i.e. $x'_i = x_i$).

$$\begin{aligned} f_{ij} &< \mathsf{cap}_{ij} \\ h'_i(\alpha) - h_i(\alpha) &< h_i(x_i) - h_i(\alpha) \\ h'_i(\alpha) &< h_i(x_i) = h'_i(x_i) \end{aligned}$$



Some properties

Based on the reassign rule the following three properties hold:

- A $h'_i(x'_i)) = \min\{h'_i(x_i), h'_i(\alpha)\}$
- B $x'_i = \alpha \neq x'_j \;\Rightarrow\; y'_{ij:\alpha} = \mathsf{cap}_{ij} + y_{ij:\alpha}$
- C $\mathsf{APF}^{\mathbf{x}',\mathbf{y}'} \leqslant \mathsf{APF}^{\mathbf{x},\mathbf{y}}$, where $\mathsf{APF}^{\mathbf{x},\mathbf{y}}$ is defined as

$$\begin{aligned} \mathsf{APF}^{\mathbf{x},\mathbf{y}} &\triangleq \sum_{i\in\mathcal{V}} h_i(x_i) = \sum_{i\in\mathcal{V}} \Big(\varphi_i(x_i) + \sum_{j\in\mathcal{V},(i,j)\in\mathcal{E}} y_{ij:x_i}\Big) \\ &= \sum_{i\in\mathcal{V}} \Big(\varphi_i(x_i) + \sum_{(i,j)\in\mathcal{E}} (y_{ij:x_i} + y_{ji:x_j})\Big) \\ &\leqslant \sum_{i\in\mathcal{V}} \varphi_i(x_i) + \sum_{(i,j)\in\mathcal{E}} w_{ij} d(x_i, x_j) = E(\mathbf{x})\ . \end{aligned}$$

The last condition shows that the algorithm terminates (assuming integer capacities), due to the reassign rule, which ensures that a new active label has always lower height than the previous active label, i.e. $h'_i(x'_i) \leqslant h_i(x_i)$.

Subroutine Update_Duals_Primals(α,x,y) *

function UPDATE_DUALS_PRIMALS(α,$\mathbf{x}$,$\mathbf{y}$)
  $\mathbf{x}' \leftarrow \mathbf{x}$, $\mathbf{y}' \leftarrow \mathbf{y}$
  Apply max-flow to G' and compute flows f_i, f_{ij}
  for all $(i,j) \in \mathcal{E}$ **do**
    $y'_{ij:\alpha} \leftarrow y_{ij:\alpha} + f_{ij} - f_{ji}$
  end for
  for all $i \in \mathcal{V}$ **do**
    $x_i \leftarrow \alpha \;\Leftrightarrow\; \exists$ unsaturated path $s \rightsquigarrow i$ in G'
  end for
  return $[\mathbf{x}', \mathbf{y}']$
end function

Subroutine PostEdit_Duals$(\alpha, \mathbf{x}', \mathbf{y}')$ *

The goal is to restore all *active balance variables* $y_{ij:x_i}$ to be non-negative.

1. $x'_i = \alpha \neq x'_j$: we have $\mathsf{cap}_{ij}, y_{ij:\alpha} \geqslant 0$, therefore $y'_{ij:\alpha} = \mathsf{cap}_{ij} + y_{ij:\alpha} \geqslant 0$.
2. $x'_i = x'_j = \alpha$: we have $y'_{ij:\alpha} = -y'_{ji:\alpha}$, therefore $\mathsf{load}'_{ij} = y'_{ij:\alpha} + y'_{ji:\alpha} = 0$. By setting $y'_{ij}(\alpha) = y'_{ji:\alpha} = 0$ we get $\mathsf{load}'_{ij} = 0$ as well.

Since none of the "load" were altered, the $\mathsf{APF}^{\mathbf{x},\mathbf{y}}$ remains unchanged.

function PostEdit_Duals$(\alpha, \mathbf{x}', \mathbf{y}')$
  for all $(i, j) \in \mathcal{E}$ with $(x'_i = x'_j = \alpha)$ and $(y'_{ij:\alpha} < 0$ or $y'_{ji:\alpha} < 0)$ **do**
    $y'_{ij:\alpha} \leftarrow 0,\ y'_{ji:\alpha} \leftarrow 0$
  end for
  for all $i \in \mathcal{V}$ **do**
    $y'_i \leftarrow \min_{\alpha \in \mathcal{L}} h'_i(\alpha)$
  end for
  return $\mathbf{y}'$
end function

ϵ_{app}-approximate solution

In summary, one can see that PD1 always leads to an ϵ-approximate solution:

Theorem 3. *The final primal-dual solutions generated by* PD1 *satisfy*

1. $h_i(x_i) = \min_{\alpha \in \mathcal{L}} h_i(\alpha)$ *for all* $i \in \mathcal{V}$,
2. $x_i \neq x_j \Rightarrow \mathsf{load}_{ij} \geqslant \frac{w_{ij} d(x_p, x_q)}{\epsilon_{app}}$ *for all* $(i, j \in \mathcal{E})$,
3. $y_{ij:\alpha} \leqslant \frac{w_{ij} d_{\min}}{2}$ *for all* $(i, j \in \mathcal{E})$ *and* $\alpha \in \mathcal{L}$,

and thus they satisfy the relaxed complementary slackness conditions with $\epsilon_1 = 1$, $\epsilon_2 = \epsilon_{app} = \frac{2 d_{\max}}{d_{\min}}$.

PD2

Parametrization of the PD2 algorithm

We now assume that d is a *metric*.

In fact, PD2 represents a family of algorithms parameterized by $\mu \in [\frac{1}{\epsilon_{\mathsf{app}}}, 1]$. Algorithm $\mathrm{PD2}_\mu$ will achieve *slackness conditions* with

$$\epsilon_1 \triangleq \mu \epsilon_{\mathsf{app}} \geqslant \frac{1}{\epsilon_{\mathsf{app}}} \epsilon_{\mathsf{app}} \geqslant 1 \quad \text{and} \quad \epsilon_2 = \epsilon_{\mathsf{app}} \ .$$

Algorithm PD1 always generates a feasible dual solution at any of its inner iterations, whereas $\mathrm{PD2}_\mu$ **may allow any such dual solution to become infeasible**.

Dual-fitting: $\mathrm{PD2}_\mu$ ensures that the (probably infeasible) final dual solution is "not too far away from feasibility", which practically means that if that solution is divided by a suitable factor, it will become feasible again.

Complementary slackness conditions

Similarly to Algorithm PD1, the equalities will hold for $i \in \mathcal{V}$

$$y_i = \min_{\alpha \in \mathcal{L}} h_i(\alpha) = h_i(x_i) = \varphi_i(x_i) + \sum_{i \in \mathcal{V}, (i,j) \in \mathcal{E}} y_{ij:x_i} \ .$$

$\mathrm{PD2}_\mu$ generates a series of intermediate pairs satisfying *complementary slackness conditions* for $\epsilon_1 \geqslant 1$ and $\epsilon_2 \geqslant \frac{1}{\mu} = \frac{1}{1/\epsilon_{\mathsf{app}}} = \epsilon_{\mathsf{app}}$:

$$\frac{\varphi_i(x_i)}{\epsilon_1} + \sum_{i \in \mathcal{V}, (i,j) \in \mathcal{E}} y_{ij:x_i} \leqslant \varphi_i(x_i) + \sum_{i \in \mathcal{V}, (i,j) \in \mathcal{E}} y_{ij:x_i} = h_i(x_i) = y_i \qquad \forall i \in \mathcal{V} \ .$$

$$\frac{w_{ij} d(x_i, x_j)}{\epsilon_2} \leqslant \mu w_{ij} d(x_i, x_j) = \mathsf{load}^{\mathbf{x},\mathbf{y}}_{ij} \qquad \forall (i,j) \in \mathcal{E} \ .$$

Like PD1, $\mathrm{PD2}_\mu$ also maintains non-negativity of *active balance variables*.

Dual fitting

The dual solution of the last intermediate pair may be infeasible, since, instead of the feasibility condition $y_{ij:\alpha} + y_{ji:\beta} \leqslant w_{ij} d(\alpha, \beta)$, $\mathrm{PD2}_\mu$ maintains the conditions:

$$y_{ij:\alpha} + y_{ji:\beta} \leqslant 2 \mu w_{ij} d_{\max} \qquad \forall (i,j) \in \mathcal{E},\ \forall \alpha, \beta \in \mathcal{L} \ .$$

These conditions also ensure that the last dual solution $\mathbf{y}$, is not "too far away from feasibility". By replacing $\mathbf{y}$ with $\mathbf{y}^{\mathsf{fit}} = \frac{\mathbf{y}}{\mu \epsilon_{\mathsf{app}}}$ we get that

$$y^{\mathsf{fit}}_{ij:\alpha} + y^{\mathsf{fit}}_{ji:\beta} = \frac{y_{ij:\alpha} + y_{ji:\beta}}{\mu \epsilon_{\mathsf{app}}} \leqslant \frac{2 \mu w_{ij} d_{\max}}{\mu \epsilon_{\mathsf{app}}} = \frac{2 \mu w_{ij} d_{\max}}{\mu 2 d_{\max} / d_{\min}} = w_{ij} d_{\min} \leqslant w_{ij} d(\alpha, \beta).$$

This means that $\mathbf{y}^{\mathsf{fit}}$ is **feasible**.

function Dual_Fit$(\mathbf{y})$
  return $\mathbf{y}^{\mathsf{fit}} \leftarrow \frac{\mathbf{y}}{\mu \epsilon_{\mathsf{app}}}$
end function

ϵ_{app}-approximate solution

The primal-dual pair $(\mathbf{x}, \mathbf{y}^{\mathsf{fit}})$ satisfies *complementary conditions* with $\epsilon_1 = \mu \epsilon_{\mathsf{app}} = 1$ and $\epsilon_2 = \epsilon_{\mathsf{app}}$ thus leading to an ϵ_{app}-approximate solution as well. Indeed, it holds that:

$$y^{\mathsf{fit}}_i \triangleq \frac{y_i}{\mu \epsilon_{\mathsf{app}}} = \frac{\min_\alpha h_i(\alpha)}{\mu \epsilon_{\mathsf{app}}} = \frac{h_i(x_i)}{\mu \epsilon_{\mathsf{app}}} = \frac{\varphi_i(x_i) + \sum_{j \in \mathcal{V}, (i,j) \in \mathcal{E}} y_{ij:x_i}}{\mu \epsilon_{\mathsf{app}}}$$
$$= \frac{\varphi_i(x_i)}{\mu \epsilon_{\mathsf{app}}} + y^{\mathsf{fit}}_{ij:x_i} \ .$$

Furthermore

$$y^{\mathsf{fit}}_{ij:x_i} + y^{\mathsf{fit}}_{ji:x_j} = \frac{y_{ij:x_i} + y_{ji:x_j}}{\mu \epsilon_{\mathsf{app}}} = \frac{\mathsf{load}_{ij}}{\mu \epsilon_{\mathsf{app}}} = \frac{\mu w_{ij} d(x_i, x_j)}{\mu \epsilon_{\mathsf{app}}} = \frac{w_{ij} d(x_i, x_j)}{\epsilon_{\mathsf{app}}} \ .$$

Update primal and dual variables

The main/only difference in the subroutine Update_Duals_Primals$(\alpha, \mathbf{x}, \mathbf{y})$ is the definition of the capacities corresponding to the *n-edges*. More precisely, assuming an α-iteration, where $x_i = \beta \neq \alpha$ and $x_j = \gamma \neq \alpha$ for a given $(i,j) \in \mathcal{E}$:

$$\mathsf{cap}_{ij} = \mu w_{ij} (d(\beta, \alpha) + d(\alpha, \gamma) - d(\beta, \gamma)) \ , \qquad (4)$$
$$\mathsf{cap}_{ji} = 0 \ .$$

All the capacities in the flow must be non-negative. This motivates that d must be a metric.

By applying $\mathsf{load}^{\mathbf{x},\mathbf{y}}_{ij} = y_{ij:\beta} + y_{ji:\gamma} = \mu w_{ij} d(\beta, \gamma)$ one can get

$$y'_{ij:\alpha} = y_{ij:\alpha} + \mathsf{cap}_{ij} = y_{ij:\alpha} + \mu w_{ij} (d(\beta, \alpha) + d(\alpha, \gamma) - d(\beta, \gamma))$$
$$= y_{ij:\alpha} + y_{ij:\beta} + y_{ji:\alpha} + \mu w_{ij} d(\alpha, \gamma) - y_{ij:\beta} - y_{ji:\gamma} = \mu w_{ij} d(\alpha, \gamma) - y_{ji:\gamma} \ ,$$

which ensures that

$$\mathsf{load}^{\mathbf{x},\mathbf{y}}_{ij} = y_{ij:\alpha} + y_{ji:\gamma} = \big(\mu w_{ij} d(\alpha, \gamma) - y_{ji:\gamma}\big) + y_{ji:\gamma} = \mu w_{ij} d(\alpha, \gamma) \ .$$

Subroutine PreEdit_Duals(α,x,y) *

The role of this routine is to edit current solution $\mathbf{y}$, before the subroutine `Update_Duals_Primals`(α,x), so that

$$\text{load}_{ij}^{\mathbf{x},\mathbf{y}} = y_{ij:\alpha} + y_{ji:\gamma} = \mu w_{ij} d(\alpha, \gamma) \ .$$

1: **function** PreEdit_Duals(α,**x**,**y**)
2: **for all** $(i,j) \in \mathcal{E}$ with $x_i \neq \alpha$, $x_j \neq \alpha$ **do**
3: $y_{ij:\alpha} \leftarrow \mu w_{ij} d(\alpha,\gamma) - y_{ji:\gamma}$
4: $y_{ji:\alpha} \leftarrow y_{ji:\gamma} - \mu w_{ij} d(\alpha,\gamma)$
5: **end for**
6: **return y**
7: **end function**

Therefore neither the "load" nor the APF function is altered.

Equivalence of PD2$_{\mu=1}$ and α-expansion

One can shown that PD2$_{\mu=1}$ indeed generates an ϵ_{app} solution.

If $\mu < 1$, then neither the primal (nor the dual) objective function necessarily decreases (increases) per iteration. Instead, APF constantly decreases.

If $\mu = 1$, then $\text{load}_{ij} = w_{ij} d(x_i, x_j)$. It can be shown that $\text{APF}^{\mathbf{x},\mathbf{y}} = E(\mathbf{x})$, whereas in any other case $\text{APF}^{\mathbf{x},\mathbf{y}} \leqslant E(\mathbf{x})$ (due to property C).

Recall that APF is the sum of active labels heights and PD2$_{\mu=1}$ always tries to choose the *lowest* label among x_p and α (see property A). During an α-iteration, PD2$_{\mu=1}$ chooses an $\mathbf{x}'$ that minimizes APF with respect to any other α-expansion $\bar{\mathbf{x}}$ of current solution $\mathbf{x}$.

Theorem 4. *Let* $(\mathbf{x}', \mathbf{y}')$ *denote the next prima-dual pair due to an* α*-iteration and* $\bar{\mathbf{x}}$ *denote* α*-expansion of the current primal. Then*

$$E(\mathbf{x}') = APF^{\mathbf{x}',\mathbf{y}'} \leqslant APF^{\bar{\mathbf{x}},\mathbf{y}'} \leqslant E(\bar{\mathbf{x}}) \ .$$

$E(\mathbf{x}') \leqslant E(\bar{\mathbf{x}})$ proves that the α-expansion algorithm is equivalent to PD2$_{\mu=1}$.

PD3

Algorithm PD3$_a$

By modifying the Algorithm PD2$_{\mu=1}$, we will get Algorithm PD3, which can be applied even if d is non-metric function.

Recall that PD2$_{\mu=1}$ maintains the *optimality criterion*: $\text{load}_{ij} \leqslant w_{ij} d(x_i, x_j)$.

Since d is not metric, we have **conflicting label-triplet** (α, β, γ):

$$d(\beta,\gamma) > d(\beta,\alpha) + d(\alpha,\gamma) \ .$$

Algorithm PD3$_a$: During the primal-dual variable update, in an α-iteration, when $x_i \neq \alpha$ and $x_j \neq \alpha$, i.e. in (4), we set $\text{cap}_{ij} = 0$.

It can be shown that for a *conflicting triplet*

$$\text{load}_{ij} = w_{ij}\big(d(\beta,\gamma) - d(\beta,\alpha)\big) \geqslant w_{ij} d(\alpha,\gamma) \ .$$

Intuitively, PD3$_a$ **overestimates the distance** between labels α, γ in order to restore the triangle inequality for the current conflicting label-triplet (α,β,γ).

PD3$_b$

We choose to set $\text{cap}_{ij} = +\infty$ and no further differences between PD3$_b$ and PD2$_{\mu=1}$ exist.

This has the following important effect: the solution $\mathbf{x}'$ produced at the current iteration, can never assign the pair of labels γ, β to the objects i, j respectively.

In the metric case we can choose the best assignment among all α-expansion moves, whereas in the non-metric case we are only able to choose the best one among a certain subset of these c-expansion moves.

PD3$_c$

PD3$_c$ first adjusts the dual solution $\mathbf{y}$ so that, for any $(i,j) \in \mathcal{E}$:

$$\text{load}_{ij} \leqslant w_{ij} d(\alpha,\gamma) + d(\gamma,\beta) \ .$$

After this initial adjustment, PD3$_c$ proceeds exactly as PD2$_{\mu=1}$, except for the fact that the term $d(\alpha,\beta)$ (4) is replaced

$$\bar{d}(\beta,\gamma) \triangleq \frac{\text{load}_{ij}}{w_{ij}} \leqslant d(\beta,\alpha) + d(\alpha,\gamma) \ .$$

Intuitively, PD3$_c$ works in a complementary way to PD3$_a$ algorithm, i.e. in order to restore the triangle inequality for the conflicting label-triplet (α,β,γ), instead of overestimating the distance between either labels α,γ or α,β, it chooses to **underestimate the distance** between labels (β,γ).

Results: stereo matching *

Original (left) PD1 PD2$_{\mu=1}$ with Potts distance

Distance $d(\alpha,\beta)$	$\epsilon_{\text{app}}^{\text{PD1}}$	$\epsilon_{\text{app}}^{\text{PD2}_{\mu=1}}$	$\epsilon_{\text{app}}^{\text{PD3}_a}$	$\epsilon_{\text{app}}^{\text{PD3}_b}$	$\epsilon_{\text{app}}^{\text{PD3}_c}$	ϵ_{app}
$[\![\alpha \neq \beta]\!]$	1.0104	1.0058	1.0058	1.0058	1.0058	2
$\min(5, \lvert\alpha-\beta\rvert)$	1.0226	1.0104	1.0104	1.0104	1.0104	10
$\min(5, \lvert\alpha-\beta\rvert)$	1.0280	-	1.0143	1.0158	1.0183	10

Literature *

- Nikos Komodakis and Georgios Tziritas. **Approximate Labeling via Graph-Cuts Based on Linear Programming**. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 29(8), pp. 1436–1453, August, 2007.
- Nikos Komodakis and Georgios Tziritas. **Approximate Labeling via the Primal-Dual Schema**. Technical Report, University of Crete, February 2005.