Variational Methods for Computer Vision: Solution Sheet 1

Exercise: 04 November 2015

Part I: Theory

1. (a) We check the individual conditions:
   - $x = y \Rightarrow d(x,y) = 0$ follows directly. Assume $\langle x-y, x-y\rangle = 0$ and $x \neq y$. Then we have $\langle z, z\rangle = 0$ for some $z = x - y \neq 0$.
   - Symmetry: $d(x,y) = \sqrt{\langle x-y, x-y\rangle} = \sqrt{\langle y-x, y-x\rangle} = d(y,x)$
   - For subadditivity, let us start with the following:

   $$\begin{aligned}\|x+y\|_2^2 &:= \langle x+y, x+y\rangle = \langle x+y, x+y\rangle \\ &= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 = (\|x\| + \|y\|)^2.\end{aligned}$$

   This implies $\|x+y\| \leq \|x\| + \|y\|$. Now we have

   $$\begin{aligned}d(x,y) &= \|x-y\| = \|x-z+z-y\| \\ &\leq \|x-z\| + \|z-y\| = d(x,z) + d(z,y).\end{aligned}$$

   (b) We verify all four conditions:
   - Summands are all positive due to the absolute value $\Rightarrow d(x,y) \geq 0$.
   - $x = y \Rightarrow d(x,y) = 0$ follows directly by substitution. For the other direction we assume $d(x,y) = 0$ which implies $|x_i - y_i| = 0$ for all $1 \leq i \leq n$, which in turn implies $x_i = y_i$.
   - Symmetry: Follows directly from symmetry of absolute value function.
   - Subadditivity: Follows directly from the basic triangle inequality ($|x+y| \leq |x| + |y|$):

   $$|x_i - z_i| = |x_i - y_i + y_i - z_i| \leq |x_i - y_i| + |y_i - z_i|.$$

   $\Rightarrow$ We have verified that the Manhattan (or Taxicab) distance is a metric. Note that the Manhattan distance is induced by the standard ℓ^1 norm on $\mathbb{R}^n$: $d(x,y) = \|x-y\|_1$.

   (c) $D(x,y) = |(x^2 - y^2)|$ is no metric, as $D(x,-x) = |(x^2 - (-x^2))| = |(x^2 - x^2)| = 0$

2. We start with the definition of the Fourier transform:

$$\begin{aligned}\mathcal{F}\{f * g\}(\nu) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y) g(x-y)\,\mathrm{d}y \right) e^{-2\pi i x \nu}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(x-y) e^{-2\pi i x \nu}\,\mathrm{d}x \right) \mathrm{d}y.\end{aligned}$$

Introducing the substitution $z = x - y$, $\mathrm{d}z = \mathrm{d}x$ we arrive at

$$\begin{aligned}\int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(x-y) e^{-2\pi i x \nu} \, \mathrm{d}x \right) \, \mathrm{d}y &= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(z) e^{-2\pi i (z+y) \nu} \, \mathrm{d}z \right) \, \mathrm{d}y \\ &= \int_{\mathbb{R}} f(y) e^{-2\pi i y \nu} \int_{\mathbb{R}} g(z) e^{-2\pi i z \nu} \, \mathrm{d}z \, \mathrm{d}y \\ &= \underbrace{\int_{\mathbb{R}} f(y) e^{-2\pi i y \nu} \, \mathrm{d}y}_{=: \mathcal{F}\{f\}(\nu)} \underbrace{\int_{\mathbb{R}} g(z) e^{-2\pi i z \nu} \, \mathrm{d}z}_{=: \mathcal{F}\{g\}(\nu)} .\end{aligned}$$

As the Fourier transform can be implemented to run in $\mathcal{O}(n \log n)$ time, convolutions can be computed efficiently by exploiting this property:

$$f * g = \mathcal{F}^{-1}\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\}.$$