Variational Methods for Computer Vision: Solution Sheet 2

Exercise: November 11, 2015

Part I: Theory

1.

$$\begin{aligned}
((f * k_1) * k_2)(x, y) &= \int \left(\int k_1(s) f(x-s, y-t)\,\mathrm{ds} \right) k_2(t)\,\mathrm{dt} \\
&= \int \int f(x-s, y-t) k_1(s) k_2(t)\,\mathrm{ds}\,\mathrm{dt} \\
&= \int \int f(x-s, y-t) \frac{1}{2\pi\sigma^2} \exp(-\frac{s^2+t^2}{2\sigma^2})\,\mathrm{ds}\,\mathrm{dt} \\
&= \int \int f(x-s, y-t) K(s,t)\,\mathrm{ds}\,\mathrm{dt} \\
&= (f * K)(x, y)
\end{aligned}$$

2. (a)

$$\begin{aligned}
\nabla \widetilde{f}(x) &= \nabla (f \circ R)(x) \\
&= (\mathrm{D}_{Rx} f \circ \mathrm{D}_x R)^T \\
&= (\mathrm{D}_{Rx} f \circ R)^T \\
&= R^T (\mathrm{D}_{Rx} f)^T \\
&= R^T \nabla f(Rx) \qquad (1)
\end{aligned}$$

Thus:

$$\begin{aligned}
R \nabla \widetilde{f}(x) &= R R^T \nabla f(Rx) \\
&= \nabla f(Rx)
\end{aligned}$$

(b)

$$\begin{aligned}
\|\nabla \widetilde{f}(x)\| &\overset{(a)}{=} \|R^T \nabla f(Rx)\| \\
&= \sqrt{\langle R^T \nabla f(Rx), R^T \nabla f(Rx) \rangle} \\
&= \sqrt{\langle \nabla f(Rx), R R^T \nabla f(Rx) \rangle} \\
&= \sqrt{\langle \nabla f(Rx), \nabla f(Rx) \rangle} \\
&= \|\nabla f(Rx)\|
\end{aligned}$$

(c)

$$\begin{aligned}
\Delta \widetilde{f}(x) &= \mathrm{div}(\underbrace{\nabla \widetilde{f}(x)}_{R^T \nabla f(Rx)}) \\
&= \mathrm{div} \begin{pmatrix} \cos(\alpha) f_x + \sin(\alpha) f_y \\ -\sin(\alpha) f_x + \cos(\alpha) f_y \end{pmatrix} = \widetilde{f}_{xx} + \widetilde{f}_{yy}
\end{aligned}$$

Thus:

$$\begin{aligned}\widetilde{f}_{xx} &= \partial_x \left(\cos(\alpha) f_x + \sin(\alpha) f_y\right) \\ &= \partial_x \cos(\alpha) f_x + \partial_x \sin(\alpha) f_y \\ &= \cos^2(\alpha) f_{xx} + \cos(\alpha)\sin(\alpha) f_{xy} + \cos(\alpha)\sin(\alpha) f_{yx} + \sin^2(\alpha) f_{yy}\end{aligned}$$

And:

$$\begin{aligned}\widetilde{f}_{yy} &= \partial_y \left(-\sin(\alpha) f_x + \cos(\alpha) f_y)\right) \\ &= \sin^2(\alpha) f_{xx} - \cos(\alpha)\sin(\alpha) f_{xy} - \cos(\alpha)\sin(\alpha) f_{yx} + \cos^2(\alpha) f_{yy}\end{aligned}$$

Therefore:

$$\begin{aligned}\widetilde{f}_{xx} + \widetilde{f}_{yy} &= f_{yy}(\underbrace{\sin^2(\alpha) + \cos^2(\alpha)}_{=1}) + f_{xx}(\underbrace{\sin^2(\alpha) + \cos^2(\alpha)}_{=1}) \\ &= \Delta\left(f(Rx)\right)\end{aligned}$$

3. (a)

$$\begin{aligned}\operatorname{div}\left(g \cdot \nabla u\right)(x) &= g\operatorname{div}\left(\nabla u\right)(x) \\ &= g\Delta u(x)\end{aligned}$$

(b)

$$\begin{aligned}\operatorname{div}\left(g\nabla u\right)(x) &= \frac{\partial}{\partial x_1}\left(g\frac{\partial}{\partial x_1}u\right)(x) + \frac{\partial}{\partial x_2}\left(g\frac{\partial}{\partial x_2}u\right)(x) \\ &= \frac{\partial^2 u}{\partial x_1^2}(x)g(x) + \frac{\partial^2 u}{\partial x_2^2}(x)g(x) + \frac{\partial g}{\partial x_1}(x)\frac{\partial u}{\partial x_1}(x) + \frac{\partial g}{\partial x_2}(x)\frac{\partial u}{\partial x_2}(x) \\ &= g(x)\Delta u(x) + \langle \nabla g(x), \nabla u(x)\rangle\end{aligned}$$