Variational Methods for Computer Vision: Exercise Sheet 4

Exercise: November 25, 2015

Part I: Theory

1. We proceed accordingly to the lecture:

$$
\begin{aligned}
\frac{\partial E(u)}{\partial u}\Big|_h &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(E(u+\varepsilon h)-E(u)\right)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^b L(u+\varepsilon h, u'+\varepsilon h', u''+\varepsilon h'') - L(u,u',u'')\,\mathrm{d}x\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^b L(u,u',u'') + \frac{\partial L}{\partial u}\varepsilon h + \frac{\partial L}{\partial u'}\varepsilon h' + \frac{\partial L}{\partial u''}\varepsilon h'' + \mathcal{O}(\varepsilon^2) - L(u,u',u'')\,\mathrm{d}x\\
&= \int_a^b \frac{\partial L}{\partial u}h + \frac{\partial L}{\partial u'}h' + \frac{\partial L}{\partial u''}h''\,\mathrm{d}x\\
&= \int_a^b \frac{\partial L}{\partial u}h\,\mathrm{d}x - \int_a^b h\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'}\,\mathrm{d}x - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u''}h'\,\mathrm{d}x + \left[h\frac{\partial L}{\partial u'}\right]_a^b + \left[h'\frac{\partial L}{\partial u''}\right]_a^b\\
&= \int_a^b h\left(\frac{\partial L}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial L}{\partial u''}\right)\,\mathrm{d}x - \left[h\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u''}\right]_a^b + \left[h\frac{\partial L}{\partial u'}\right]_a^b + \left[h'\frac{\partial L}{\partial u''}\right]_a^b.
\end{aligned}
$$

2.

$$
\begin{aligned}
\frac{\partial E(u)}{\partial u}\Big|_h &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(E(u+\varepsilon h)-E(u)\right)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega L(u+\varepsilon h, \nabla(u+\varepsilon h)) - L(u,\nabla u)\,\mathrm{d}x\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega L(u,\nabla u) + \frac{\partial L}{\partial u}\varepsilon h + \frac{\partial L}{\partial u_x}\varepsilon\frac{\partial h}{\partial x} + \frac{\partial L}{\partial u_y}\varepsilon\frac{\partial h}{\partial y} + \frac{\partial L}{\partial u_z}\varepsilon\frac{\partial h}{\partial z} + \mathcal{O}(\varepsilon^2) - L(u,\nabla u)\,\mathrm{d}x\\
&= \int_\Omega \frac{\partial L}{\partial u}h\,\mathrm{d}x + \int_\Omega \left\langle \nabla h, \begin{pmatrix}\frac{\partial L}{\partial u_x} & \frac{\partial L}{\partial u_y} & \frac{\partial L}{\partial u_z}\end{pmatrix}^T\right\rangle\,\mathrm{d}x\\
&= \int_\Omega \frac{\partial L}{\partial u}h\,\mathrm{d}x - \int_\Omega \left\langle h, \operatorname{div}\begin{pmatrix}\frac{\partial L}{\partial u_x} & \frac{\partial L}{\partial u_y} & \frac{\partial L}{\partial u_z}\end{pmatrix}^T\right\rangle\,\mathrm{d}x\\
&\qquad + \int_{\partial\Omega} h\left\langle \begin{pmatrix}\frac{\partial L}{\partial u_x} & \frac{\partial L}{\partial u_y} & \frac{\partial L}{\partial u_z}\end{pmatrix}^T, n\right\rangle\mathrm{d}\partial\Omega\\
&= \int_\Omega h\left(\frac{\partial L}{\partial u} - \left(\frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial L}{\partial u_y} + \frac{\partial}{\partial z}\frac{\partial L}{\partial u_z}\right)\right)\mathrm{d}x\\
&\qquad + \int_{\partial\Omega} h\left(\frac{\partial L}{\partial u_x}n_x + \frac{\partial L}{\partial u_y}n_y + \frac{\partial L}{\partial u_z}n_z\right)\mathrm{d}\partial\Omega.
\end{aligned}
$$

Hence we can write the Euler-Lagrange equation as

$$\begin{aligned} \frac{\partial L}{\partial u} - \operatorname{div}\left(\frac{\partial L}{\partial \nabla u}\right) &= 0, && \text{on } \Omega, \\ \left\langle \frac{\partial L}{\partial \nabla u}, \nu \right\rangle &= 0, && \text{on } \partial\Omega, \end{aligned}$$

where ν denotes the normal vector on the boundary $\partial\Omega$.

3. (a) We have that $L(u, \nabla u) = \sqrt{u_x^2 + u_y^2}$ and

$$\begin{aligned} \frac{\partial L}{\partial u_x} &= \frac{u_x}{\sqrt{u_x^2 + u_y^2}}, \\ \frac{\partial L}{\partial u_y} &= \frac{u_y}{\sqrt{u_x^2 + u_y^2}}. \end{aligned}$$

Thus the Euler-Lagrange equation is given as the following:

$$\frac{\partial E}{\partial u} = \frac{\partial L}{\partial u} - \operatorname{div}\left(\frac{\partial L}{\partial \nabla u}\right) = -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

(b) Similarly to the previous exercise we arrive for $L(u, \nabla u) = \sqrt{(\nabla u)^\mathsf{T} D \nabla u}$ at the following Euler-Lagrange equation:

$$\frac{\partial E}{\partial u} = \frac{\partial L}{\partial u} - \operatorname{div}\left(\frac{\partial L}{\partial \nabla u}\right) = -\operatorname{div}\left(\frac{(D + D^\mathsf{T})\nabla u}{2\sqrt{(\nabla u)^\mathsf{T} D \nabla u}}\right).$$