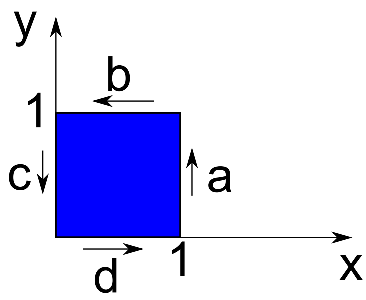
Variational Methods for Computer Vision: Solution Sheet 6

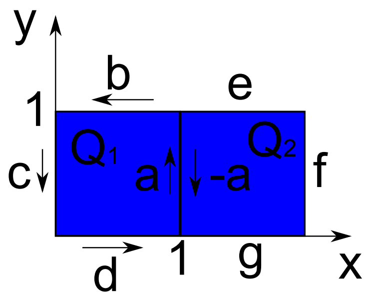
Exercise: 09 December 2015

Part I: Theory



1. (a)

$$
\begin{aligned}
\int_Q v_x(x,y) - u_y(x,y)\mathrm{dxdy} &= \int_0^1 \int_0^1 v_x(x,y) - u_y(x,y)\mathrm{dxdy} \\
&= \int_0^1 \int_0^1 v_x(x,y)\mathrm{dxdy} - \int_0^1 \int_0^1 u_y(x,y)\mathrm{dydx} \\
&= \int_0^1 \left. v(x,y)\right|_{x=0}^{x=1} \mathrm{dy} - \int_0^1 \left. v(x,y)\right|_{y=0}^{y=1} \mathrm{dx} \\
&= \int_0^1 v(1,y) - v(0,y)\mathrm{dy} - \int_0^1 v(x,1) - v(x,0)\mathrm{dx} \\
&= \int_0^1 v(1,y)\mathrm{dy} - \int_0^1 v(0,y)\mathrm{dy} - \int_0^1 v(x,1)\mathrm{dx} + \int_0^1 v(x,0)\mathrm{dx} \\
&= \underbrace{\int_0^1 v(1,y)\mathrm{dy}}_{\int_a v(x,y)\mathrm{dy}} + \underbrace{\int_1^0 v(0,y)\mathrm{dy}}_{\int_c v(x,y)\mathrm{dy}} + \underbrace{\int_1^0 v(x,1)\mathrm{dx}}_{\int_b u(x,y)\mathrm{dx}} + \underbrace{\int_0^1 v(x,0)\mathrm{dx}}_{\int_d v(x,y)\mathrm{dx}} \\
&= \int_{\partial Q} v\mathrm{ds}
\end{aligned}
$$



(b)

$$
\begin{aligned}
&\int_{Q_1} v_x(x,y) - u_y(x,y)\mathrm{dxdy} + \int_{Q_2} v_x(x,y) - u_y(x,y)\mathrm{dxdy} \\
=&\int_a v(x,y)\mathrm{dy} \int_c v(x,y)\mathrm{dy} + \int_b v(x,y)\mathrm{dx} + \int_d v(x,y)\mathrm{dx} \\
-&\int_a v(x,y)\mathrm{dy} + \int_e v(x,y)\mathrm{dx} + \int_f v(x,y)\mathrm{dy} + \int_g v(x,y)\mathrm{dx}
\end{aligned}
$$

2. (a) The curvature κ of a circle with radius r is $\kappa = \frac{1}{r}$. We can use this fact in calculating the Euler-Lagrange equations for the 2 different cases.

$r > 1$:

$$
\begin{aligned}
u_{\text{outer}} &= 0 \\
u_{\text{inner}} &= \frac{\pi}{\pi r^2} = \frac{1}{r^2}
\end{aligned}
$$

This leads to following Euler-Lagrange equation:

$$
\begin{aligned}
&(I - u_{\text{outer}})^2 - (I - u_{\text{inner}})^2 - \nu\kappa \\
=&(0-0)^2 - (0 - \frac{1}{r^2})^2 - \frac{\nu}{r} \\
=&-\frac{1}{r^2} - \frac{\nu}{r}
\end{aligned}
$$

$r \leq 1$:

$$
\begin{aligned}
u_{\text{outer}} &= \frac{\pi - \pi r^2}{100 - \pi r^2} \\
u_{\text{inner}} &= 1
\end{aligned}
$$

This leads to following Euler-Lagrange equation:

$$
\begin{aligned}
&(I - u_{\text{outer}})^2 - (I - u_{\text{inner}})^2 - \nu\kappa \\
=&\left(1 - \frac{\pi - \pi r^2}{100 - \pi r^2}\right)^2 - 0 - \frac{\nu}{r} \\
=&\frac{100 - \pi r^2 - \pi - \pi r^2}{100 - \pi r^2} - \frac{\nu}{r} \\
=&\left(\frac{100 - \pi}{100 - \pi r^2}\right)^2 - \frac{\nu}{r}
\end{aligned}
$$

(b)

$$\lim_{r \searrow 1} -\frac{1}{r^2} - \frac{\nu}{r} = -1 - \nu$$
$$\lim_{r \nearrow 1} -\frac{100-\pi}{100-\pi r^2} - \frac{\nu}{r} = \frac{100-\pi}{100-\pi} - \nu = 1 - \nu$$

As the limits differ the Gateaux derivative at $r = 1$ is not continuous.

$\nu \leq 1$ is a good choice because it ensures that the curve evolves in the right direction for both cases $r > 1$ and $r \leq 1$.

$r > 1$:

$$\nu \leq 1 \Rightarrow -\frac{1}{r^2} - \frac{\nu}{r} < 0$$

$r \leq 1$:

$$\nu \leq 1 \Rightarrow \left(\frac{100-\pi}{100-\pi r^2}\right)^2 - \frac{\nu}{r} \geq 0$$