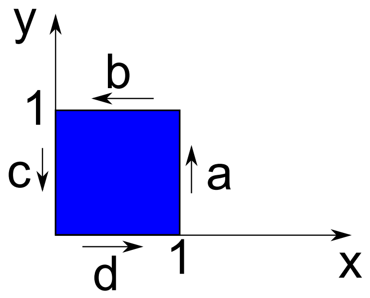
Variational Methods for Computer Vision: Solution Sheet 6

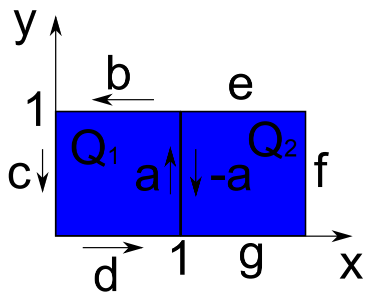
Exercise: 09 December 2015

Part I: Theory



1. (a)

$$
\begin{aligned}
\int_Q v_x(x,y) - u_y(x,y)\mathrm{dxdy} &= \int_0^1 \int_0^1 v_x(x,y) - u_y(x,y)\mathrm{dxdy} \\
&= \int_0^1 \int_0^1 v_x(x,y)\mathrm{dxdy} - \int_0^1 \int_0^1 u_y(x,y)\mathrm{dydx} \\
&= \int_0^1 v(x,y)\big|_{x=0}^{x=1}\,\mathrm{dy} - \int_0^1 v(x,y)\big|_{y=0}^{y=1}\,\mathrm{dx} \\
&= \int_0^1 v(1,y) - v(0,y)\mathrm{dy} - \int_0^1 v(x,1) - v(x,0)\mathrm{dx} \\
&= \int_0^1 v(1,y)\mathrm{dy} - \int_0^1 v(0,y)\mathrm{dy} - \int_0^1 v(x,1)\mathrm{dx} + \int_0^1 v(x,0)\mathrm{dx} \\
&= \underbrace{\int_0^1 v(1,y)\mathrm{dy}}_{\int_a v(x,y)\mathrm{dy}} + \underbrace{\int_1^0 v(0,y)\mathrm{dy}}_{\int_c v(x,y)\mathrm{dy}} + \underbrace{\int_1^0 v(x,1)\mathrm{dx}}_{\int_b u(x,y)\mathrm{dx}} + \underbrace{\int_0^1 v(x,0)\mathrm{dx}}_{\int_d v(x,y)\mathrm{dx}} \\
&= \int_{\partial Q} v\mathrm{ds}
\end{aligned}
$$



(b)

$$\int_{Q_1} v_x(x,y) - u_y(x,y)\mathrm{dxdy} + \int_{Q_2} v_x(x,y) - u_y(x,y)\mathrm{dxdy}$$
$$= \int_a v(x,y)\mathrm{dy} \int_c v(x,y)\mathrm{dy} + \int_b v(x,y)\mathrm{dx} + \int_d v(x,y)\mathrm{dx}$$
$$- \int_a v(x,y)\mathrm{dy} + \int_e v(x,y)\mathrm{dx} + \int_f v(x,y)\mathrm{dy} + \int_g v(x,y)\mathrm{dx}$$

2. Consider the energies of regions Ω_1 and Ω_2 *before* and *after* the merge operation:

$$E_{\text{before}} = \sum_{x\in\Omega_1} (I(x) - u_1)^2 + \sum_{x\in\Omega_2} (I(x) - u_2)^2 + \nu|C_{\text{before}}|$$
$$E_{\text{after}} = \sum_{x\in\Omega_1\cup\Omega_2} (I(x) - u_{\text{merged}})^2 + \nu|C_{\text{after}}|$$

So the change in energy δE becomes:

$$\delta E = E_{\text{after}} - E_{\text{before}}$$
$$= \sum_{x\in\Omega_1\cup\Omega_2} (I(x) - u_{\text{merged}})^2 - \sum_{x\in\Omega_1} (I(x) - u_1)^2 - \sum_{x\in\Omega_2} (I(x) - u_2)^2 - \nu\delta C \qquad (*)$$
$$= \sum_{x\in\Omega_1\cup\Omega_2} I(x) - (A_1 + A_2)u_{\text{merged}}^2 - \sum_{x\in\Omega_1} I(x)^2 + A_1{u_1}^2 - \sum_{x\in\Omega_2} I(x)^2 + A_2{u_2}^2 - \nu\delta C \qquad (**)$$
$$= A_1{u_1}^2 + A_2{u_2}^2 - (A_1 + A_2)\left(\frac{u_1A_1 + u_2A_2}{A_1 + A_2}\right)^2 - \nu\delta C$$
$$= A_1{u_1}^2 + A_2{u_2}^2 - \frac{(u_1A_1 + u_2A_2)^2}{A_1 + A_2} - \nu\delta C$$
$$= A_1{u_1}^2 + A_2{u_2}^2 - \frac{(u_1A_1)^2 + 2u_1A_1u_2A_2 + (u_2A_2)^2}{A_1 + A_2} - \nu\delta C$$
$$= \frac{(A_1 + A_2)A_1{u_1}^2 + (A_1 + A_2)A_2{u_2}^2 - (u_1A_1)^2 - 2u_1A_1u_2A_2 - (u_2A_2)^2}{A_1 + A_2} - \nu\delta C$$
$$= \frac{A_1A_2{u_1}^2 + A_1A_2{u_2}^2 - 2A_1A_2u_1u_2}{A_1 + A_2} - \nu\delta C$$
$$= \frac{A_1A_2}{A_1 + A_2}({u_1}^2 - {u_2}^2) - \nu\delta C$$

?? Using the identity $\sum_{i=1}^{n}(x_i - \overline{x}) = \sum_{i=1}^{n} {x_i}^2 - n\overline{x}^2$

?? Using $u_{\text{merged}} = \frac{u_1A_1+u_2A_2}{A_1+A_2}$

3. (a) The curvature κ of a circle with radius r is $\kappa = \frac{1}{r}$. We can use this fact in calculating the Euler-Lagrange equations for the 2 different cases.

$r > 1$:

$$\begin{aligned} u_{\text{outer}} &= 0 \\ u_{\text{inner}} &= \frac{\pi}{\pi r^2} = \frac{1}{r^2} \end{aligned}$$

This leads to following Euler-Lagrange equation:

$$\begin{aligned} &(I - u_{\text{outer}})^2 - (I - u_{\text{inner}})^2 - \nu\kappa \\ =&(0-0)^2 - (0 - \frac{1}{r^2})^2 - \frac{\nu}{r} \\ =& -\frac{1}{r^2} - \frac{\nu}{r} \end{aligned}$$

$r \leq 1$:

$$\begin{aligned} u_{\text{outer}} &= \frac{\pi - \pi r^2}{100 - \pi r^2} \\ u_{\text{inner}} &= 1 \end{aligned}$$

This leads to following Euler-Lagrange equation:

$$\begin{aligned} &(I - u_{\text{outer}})^2 - (I - u_{\text{inner}})^2 - \nu\kappa \\ =&\left(1 - \frac{\pi - \pi r^2}{100 - \pi r^2}\right)^2 - 0 - \frac{\nu}{r} \\ =&\frac{100 - \pi r^2 - \pi - \pi r^2}{100 - \pi r^2} - \frac{\nu}{r} \\ =&\left(\frac{100 - \pi}{100 - \pi r^2}\right)^2 - \frac{\nu}{r} \end{aligned}$$

(b)

$$\begin{aligned} &\lim_{r \searrow 1} -\frac{1}{r^2} - \frac{\nu}{r} = -1 - \nu \\ &\lim_{r \nearrow 1} -\frac{100 - \pi}{100 - \pi r^2} - \frac{\nu}{r} = \frac{100 - \pi}{100 - \pi} - \nu = 1 - \nu \end{aligned}$$

As the limits differ the Gateux derivative at $r = 1$ is not continuous.

$\nu \leq 1$ is a good choice because it ensures that the curve evolves in the right direction for both cases $r > 1$ and $r \leq 1$.

$r > 1$:

$$\nu \leq 1 \Rightarrow -\frac{1}{r^2} - \frac{\nu}{r} < 0$$

$r \leq 1$:

$$\nu \leq 1 \Rightarrow \left(\frac{100 - \pi}{100 - \pi r^2}\right)^2 - \frac{\nu}{r} \geq 0$$