Exercise: January 13, 2016

Part I: Theory

1. The energy before and after the merging are given as:

$$E_{before} = \int_{\Omega_1} (I(x) - u_1)^2 dx + \int_{\Omega_2} (I(x) - u_2)^2 dx + \nu |C_{before}|,$$

$$E_{after} = \int_{\Omega_1 \cup \Omega_2} (I(x) - u_{merged})^2 dx + \nu |C_{after}|.$$
(1)

Furthermore, we will make use of the following identites:

$$\int_{\Omega} (f(x) - \bar{f})^2 dx = \int_{\Omega} f(x)^2 dx - |\Omega| \bar{f}^2,$$

$$u_{merged} = \frac{u_1 A_1 + u_2 A_2}{A_1 + A_2}.$$
(2)

A straightforward calculation yields the desired result:

$$\begin{split} \delta E &= E_{after} - E_{before} \\ &= \int_{\Omega_1 \cup \Omega_2} \left(I(x) - u_{merged} \right)^2 - \int_{\Omega_1} \left(I(x) - u_1 \right)^2 dx - \int_{\Omega_2} \left(I(x) - u_2 \right)^2 dx - \nu \delta C \\ &= \int_{\Omega_1 \cup \Omega_2} I(x)^2 dx - (A_1 + A_2) u_{merged}^2 \\ &- \int_{\Omega_1} I(x)^2 dx + A_1 u_1^2 - \int_{\Omega_2} I(x)^2 dx + A_2 u_2^2 - \nu \delta C \\ &= -(A_1 + A_2) \left(\frac{u_1 A_1 + u_2 A_2}{A_1 + A_2} \right)^2 + A_1 u_1^2 + A_2 u_2^2 - \nu \delta C \\ &= \frac{A_1 A_2}{A_1 + A_2} (u_1^2 - u_2^2) - \nu \delta C. \end{split}$$
(3)