

Variational Methods for Computer Vision: Solution Sheet 8

Exercise: 21 January 2015

Part I: Theory

1. Using the property that the delta distribution $\delta(\phi)$ is the derivative of H , we get:

$$|\nabla H(\phi)| = |\delta(\phi) \cdot \nabla \phi(x)| = \delta(\phi) |\nabla \phi(x)|$$

So we can rewrite the functional in the following form:

$$\begin{aligned}\mathcal{L} &= f_1 \cdot H(\phi) + f_2(1 - H(\phi)) + \nu |\nabla H(\phi)| \\ &= (f_1 - f_2)H(\phi) + \nu \delta(\phi) |\nabla \phi(x)|\end{aligned}$$

So:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \nabla \phi} &= \frac{\partial}{\partial \nabla \phi} \nu |\nabla \phi| \delta(\phi) \\ &= \nu \cdot \delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|}\end{aligned}$$

And:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi} &= H'(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi(x)| \\ &= \delta(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi(x)|\end{aligned}$$

Using both results we can compute

$$\begin{aligned}\frac{\partial E}{\partial \phi} &= \frac{\partial \mathcal{L}}{\partial \phi} - \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial \nabla \phi} \right) \\ &= \delta(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi(x)| - \nu \operatorname{div} \left(\delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|} \right)\end{aligned}$$

Further simplifying gives:

$$\begin{aligned}\operatorname{div} \left(\delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|} \right) &= \frac{d}{dx} \left(\delta(\phi) \cdot \frac{\nabla \phi}{|\nabla(\phi)|} \right) + \frac{d}{dy} \left(\delta(\phi) \cdot \frac{\nabla \phi}{|\nabla(\phi)|} \right) \\ &= \left(\frac{d}{dx} \delta(\phi) \right) \cdot \frac{\nabla \phi}{|\nabla(\phi)|} + \left(\frac{d}{dx} \frac{\nabla \phi}{|\nabla(\phi)|} \right) \cdot \delta(\phi) \\ &\quad + \left(\frac{d}{dy} \delta(\phi) \right) \cdot \frac{\nabla \phi}{|\nabla(\phi)|} + \left(\frac{d}{dy} \frac{\nabla \phi}{|\nabla(\phi)|} \right) \cdot \delta(\phi) \\ &= \delta'(\phi) \cdot \phi_x \cdot \frac{\nabla \phi}{|\nabla(\phi)|} + \left(\frac{d}{dx} \frac{\nabla \phi}{|\nabla(\phi)|} \right) \cdot \delta(\phi) \\ &\quad + \delta'(x) \cdot \phi_y \cdot \frac{\nabla \phi}{|\nabla(\phi)|} + \left(\frac{d}{dy} \frac{\nabla \phi}{|\nabla(\phi)|} \right) \cdot \delta(\phi) \\ &= \delta(\phi) \operatorname{div} \left(\frac{\nabla \phi}{|\nabla(\phi)|} \right) + \delta'(\phi) \cdot \nabla \phi \cdot \frac{\nabla \phi}{|\nabla(\phi)|} \\ &= \delta'(\phi) |\nabla \phi| + \delta(\phi) \cdot \operatorname{div} \left(\frac{\nabla \phi}{|\nabla(\phi)|} \right)\end{aligned}$$

Finally:

$$\begin{aligned}\frac{\partial E}{\partial \phi} &= \delta(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi(x)| - \nu \delta'(\phi) |\nabla \phi| - \nu \delta(\phi) \cdot \operatorname{div} \left(\frac{\nabla \phi}{|\nabla(\phi)|} \right) \\ &= \delta \phi \left(f_1 - f_2 - \nu \operatorname{div} \left(\frac{\nabla \phi}{|\nabla(\phi)|} \right) \right)\end{aligned}$$

2.

$$\begin{aligned}E(\phi) &= \int g(x) |\nabla H(\phi(x))| dx \\ &= \int g(x) \delta(\phi) |\nabla \phi| dx\end{aligned}$$

With $\frac{\partial E}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial \nabla \phi} \right)$ this leads to:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \delta'(\phi) g(x) |\nabla \phi|$$

and

$$\frac{\partial \mathcal{L}}{\partial \nabla \phi} = g(x) \delta(\phi) \frac{\nabla \phi}{|\nabla(\phi)|}$$

Combining both results and further evaluating $\operatorname{div} \left(g(x) \delta(\phi) \frac{\nabla \phi}{|\nabla(\phi)|} \right)$ gives:

$$\begin{aligned}\frac{\partial E}{\partial \phi} &= \frac{\partial L}{\partial \phi} - \operatorname{div} \left(\frac{\partial L}{\partial \nabla \phi} \right) \\ &= \delta'(\phi) g(x) |\nabla \phi| - \operatorname{div} \left(g(x) \delta(\phi) \frac{\nabla \phi}{|\nabla \phi|} \right) \\ &= \delta'(\phi) g(x) |\nabla \phi| - \delta'(\phi) g(x) \frac{\nabla \phi \cdot \nabla \phi}{|\nabla \phi|} - \delta(\phi) \operatorname{div} \left(g(x) \frac{\nabla \phi}{|\nabla \phi|} \right) \\ &= -\delta(\phi) \operatorname{div} \left(g(x) \frac{\nabla \phi}{|\nabla \phi|} \right)\end{aligned}$$