IN2107 - Image Segmentation and Shape Analysis (Seminar)

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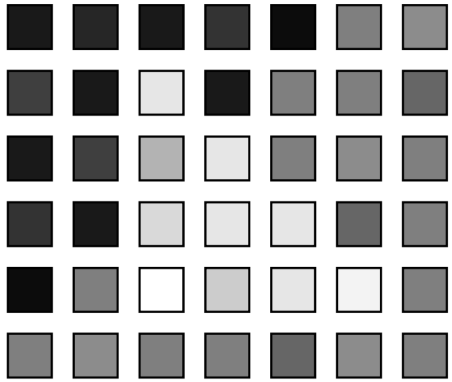
Winter Semester 2016/2017

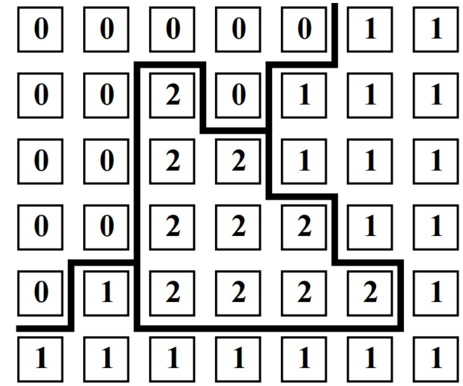
An Experimental Comparison of Min-Cut/Max-Flow Algorithms for Energy Minimization in Vision

Energy minimization

The labeling problem



An input image



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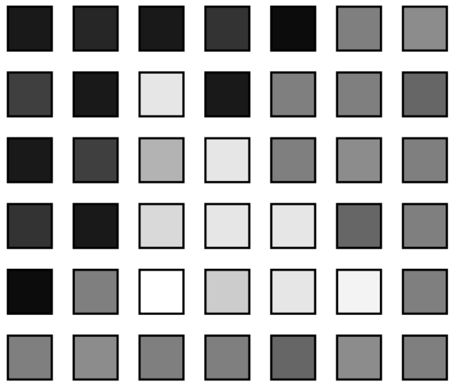

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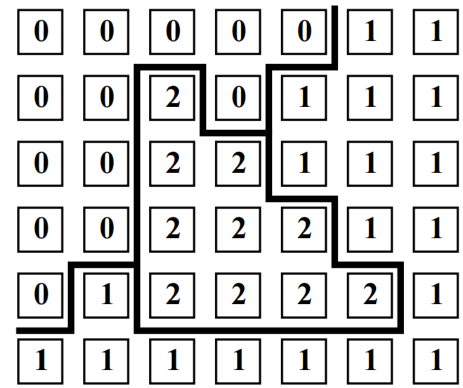

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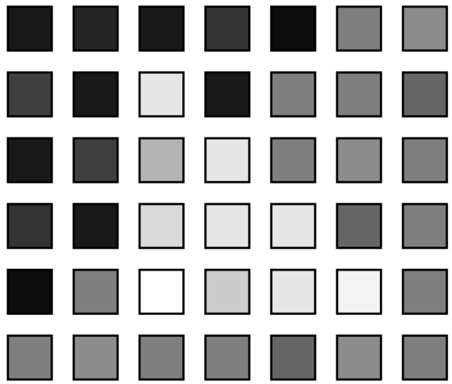

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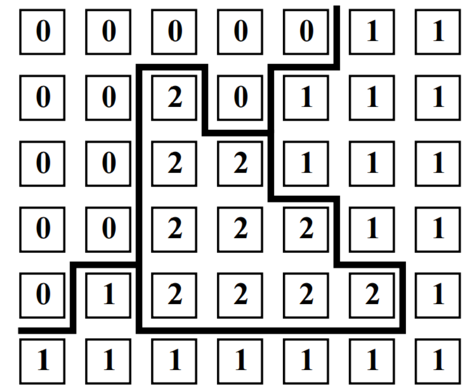

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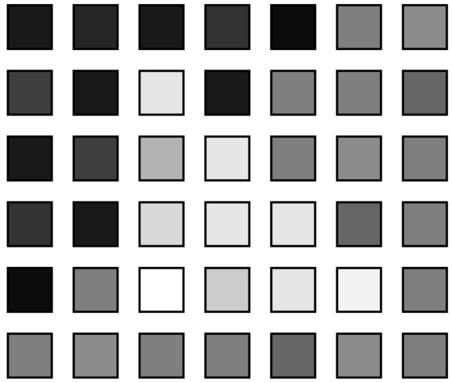

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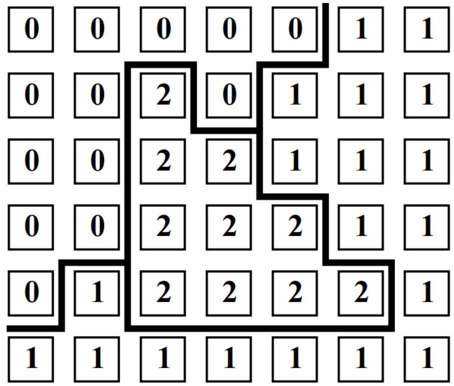

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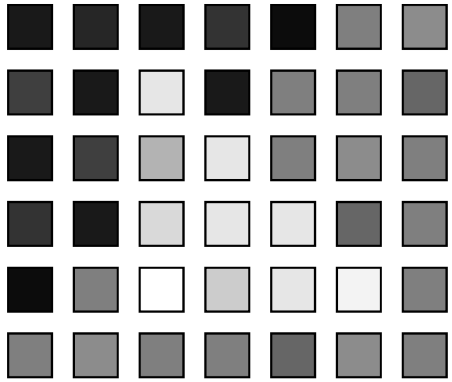

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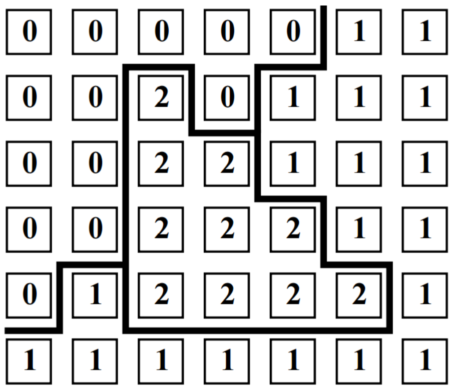

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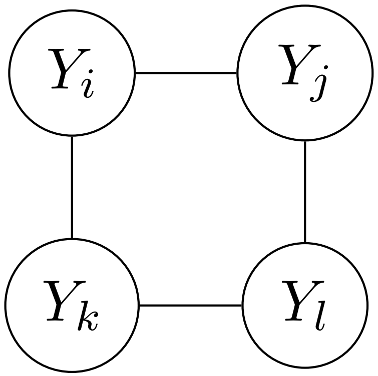
We aim to model the joint probability distribution $p(\mathbf{y})$, and find the *best* labeling $\mathbf{y}^*$

Markov random field

Consider an undirected graph $G = (\mathcal{V}, \mathcal{E})$ with the following assumption:

Two nodes (i.e. random variables) are *conditionally independent* whenever they are not connected, that is for any node i in the graph

$$p(Y_i \mid Y_{\mathcal{V}\setminus\{i\}}) = p(Y_i \mid Y_{N(i)}) \ ,$$



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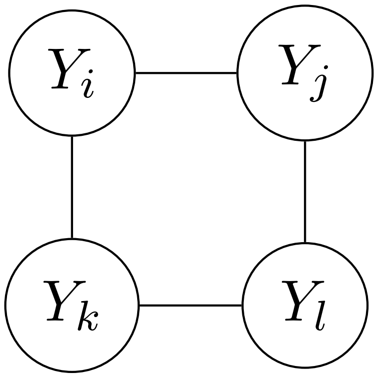

A *probability distribution* $p(\mathbf{y})$ is called *Gibbs distribution* if it can be factorized into potential functions $\psi_c(\mathbf{y}_c) > 0$ defined on cliques:

$$p(\mathbf{y}) = \frac{1}{Z} \prod_{c \in \mathcal{C}_G} \psi_c(\mathbf{y}_c) \ , \text{ where } Z = \sum_{\mathbf{y} \in \mathcal{Y}} \prod_{c \in \mathcal{C}_G} \psi_c(\mathbf{y}_c) \ ,$$

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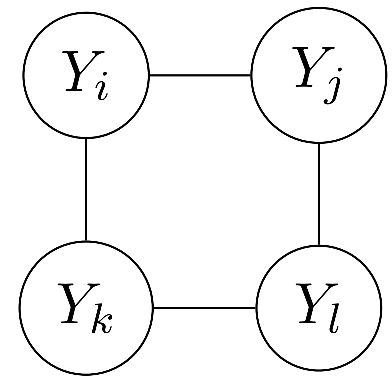

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The *Hammersley-Clifford theorem* tells us that the above two definitions are equivalent.

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Assuming $\psi_c : \mathcal{Y}_c \rightarrow \mathbb{R}^+$, where $\mathcal{Y}_c = \times_{i \in N(c)} \mathcal{Y}_i$ is the product domain of the clique c, instead of *potentials*, we can also work with *energies*

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Hence, $p(\mathbf{y})$ is completely determined by $E(\mathbf{y})$

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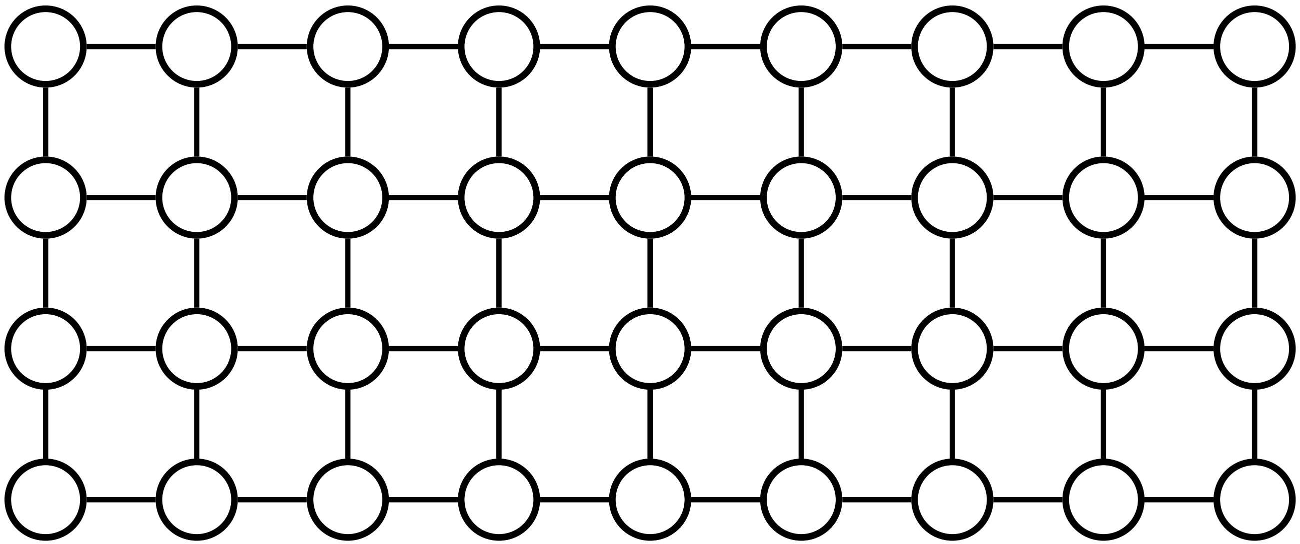
Energy



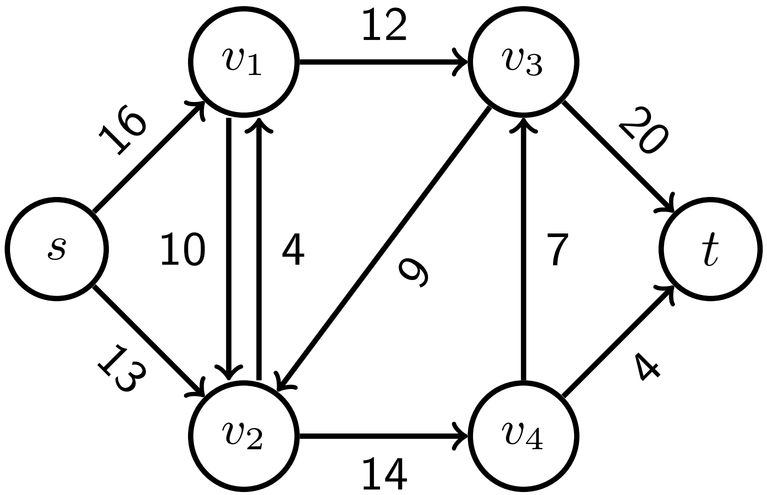
In practice, one typically models the energy function directly

$$E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j \in \mathcal{E})} E_{ij}(y_i, y_j)$$

minCut/maxFlow

Graph cut

Assume a *weighted directed graph* $G = (\mathcal{V}, \mathcal{E}, c)$



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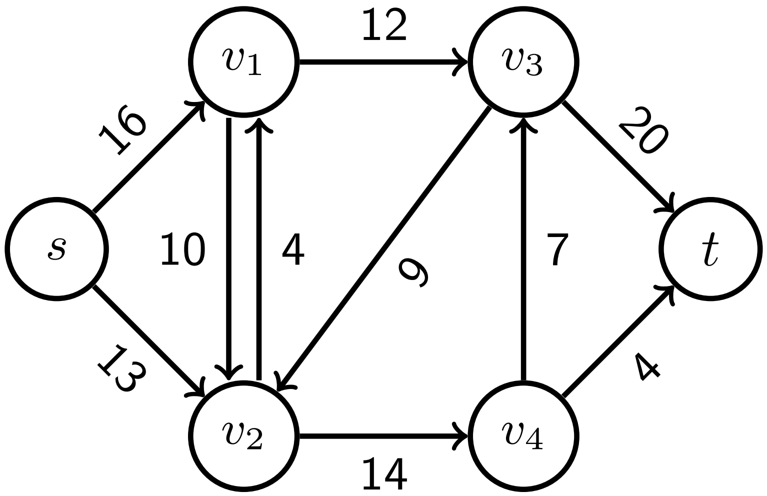

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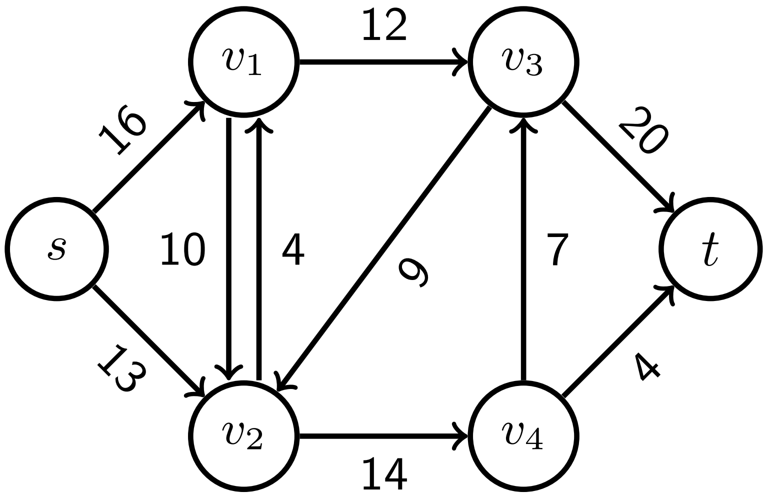

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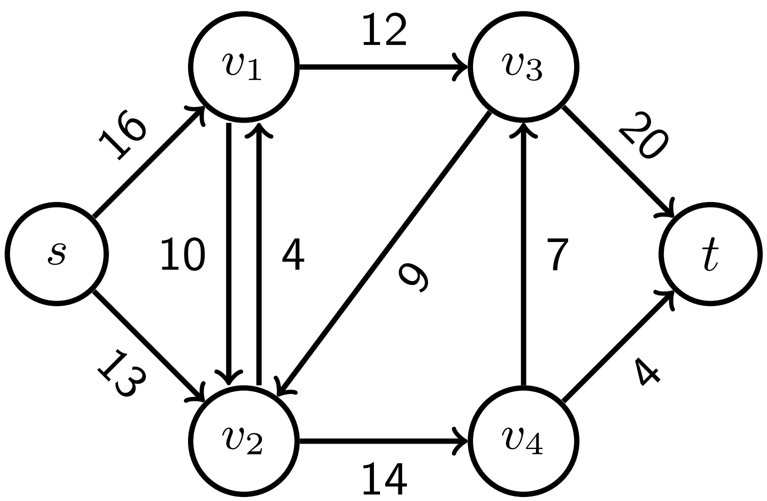

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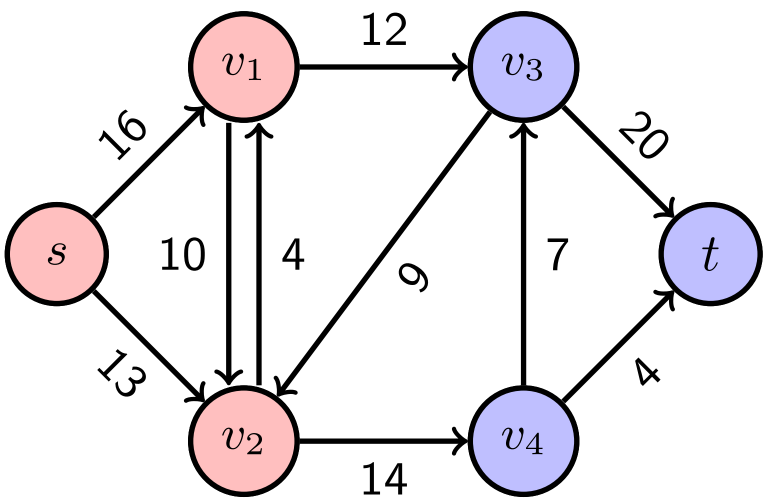

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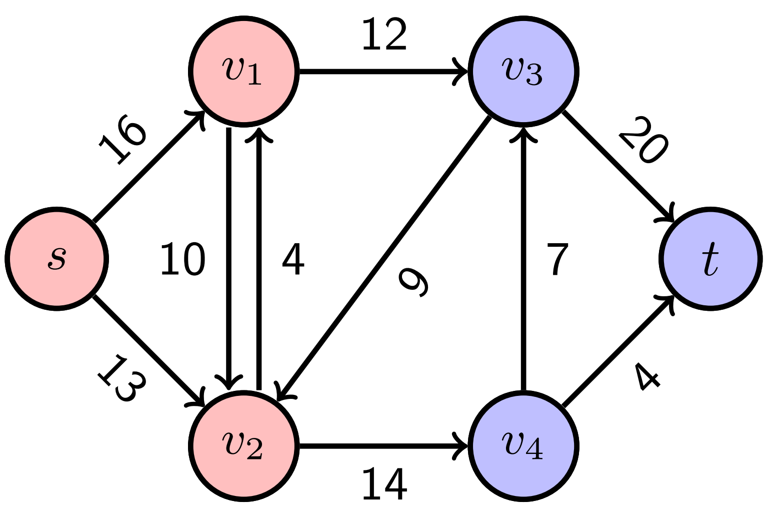

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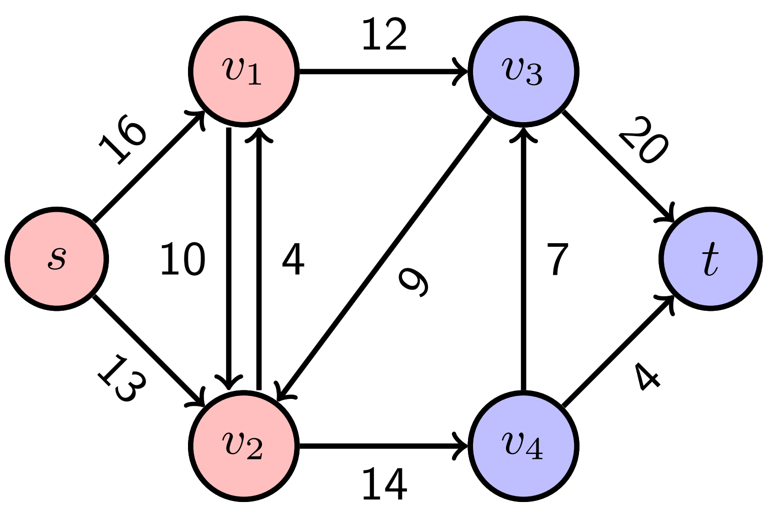

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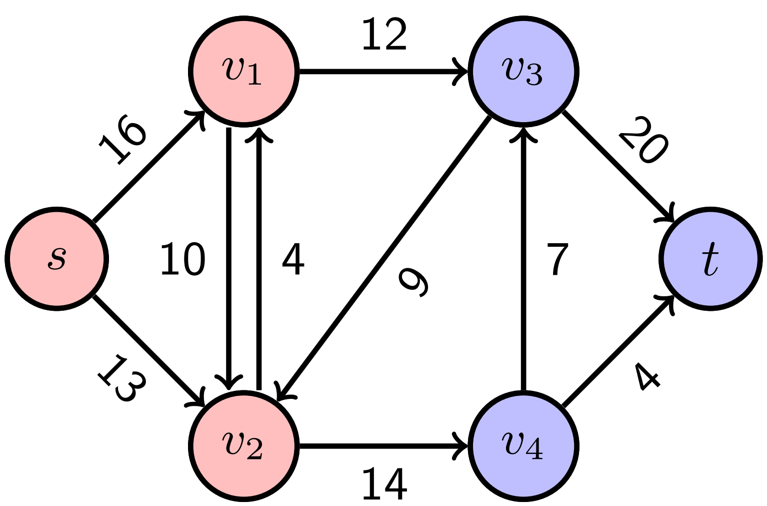

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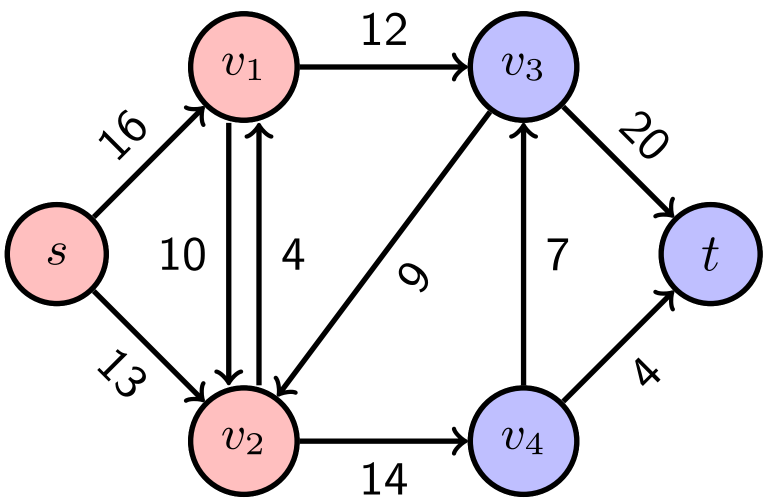

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<u>*Example*</u>: $\text{cut}(\mathcal{S}, \mathcal{T}) = c(v_1, v_3) + c(v_2, v_4) = 12 + 14 = 26$.

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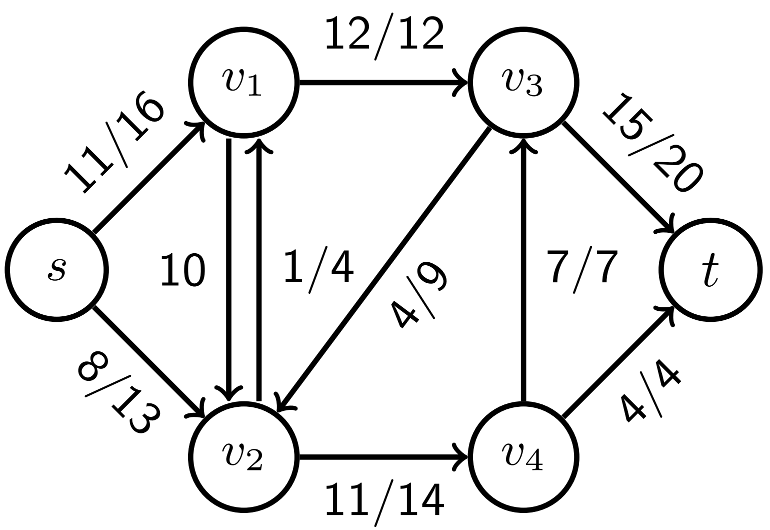
1. $f(i,j) \leqslant c(i,j)$ for all $(i,j) \in \mathcal{E}$.
2. For all $i \in \mathcal{V} \backslash \{s,t\}$

$$\sum_{(i,j)\in\mathcal{E}} f(i,j) = \sum_{(j,i)\in\mathcal{E}} f(j,i) \, .$$



The edges are labeled by $f(i,j)/c(i,j)$.

Only positive $f(i,j)$ are shown.

The value of a flow

The *value* of a flow f is defined as

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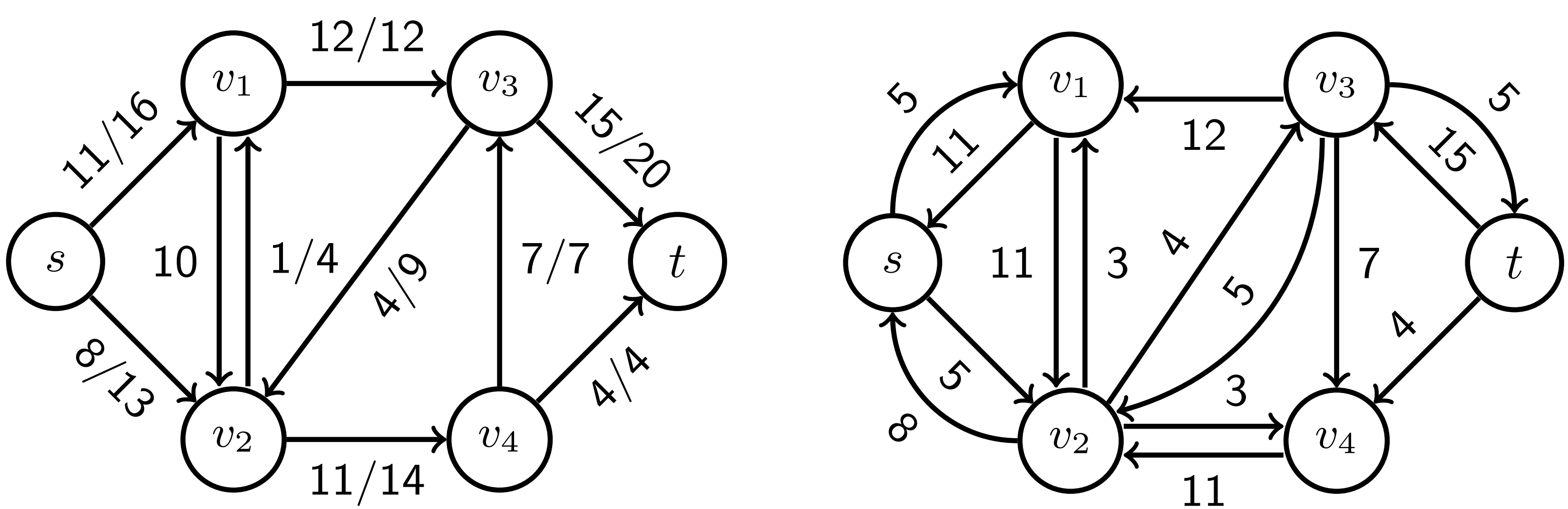
The *maximum-flow problem* is to find a *flow* f with the highest cost for a given flow network G

Residual network

Let $G = (\mathcal{V}, \mathcal{E}, c, s, t)$ be a *flow network* and let f be a *flow* in G. The *weighted directed graph* $G_f = (\mathcal{V}, \mathcal{E}_f, c_f)$ is called *residual network* of G induced by f, where



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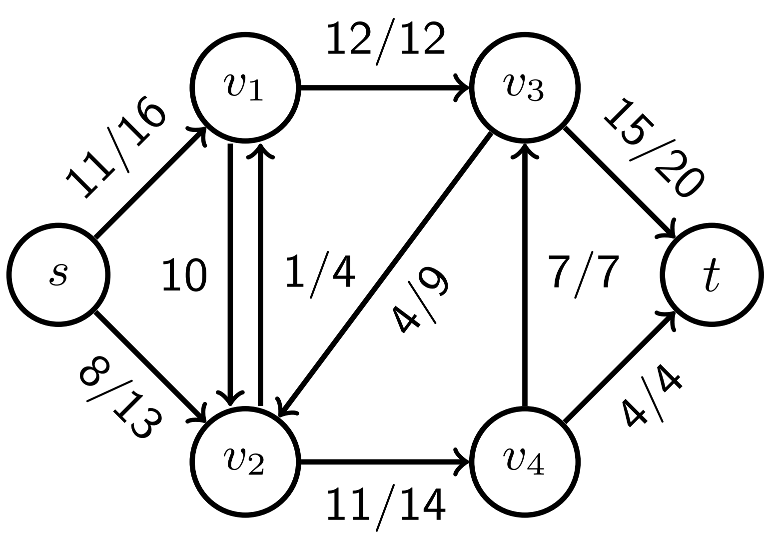

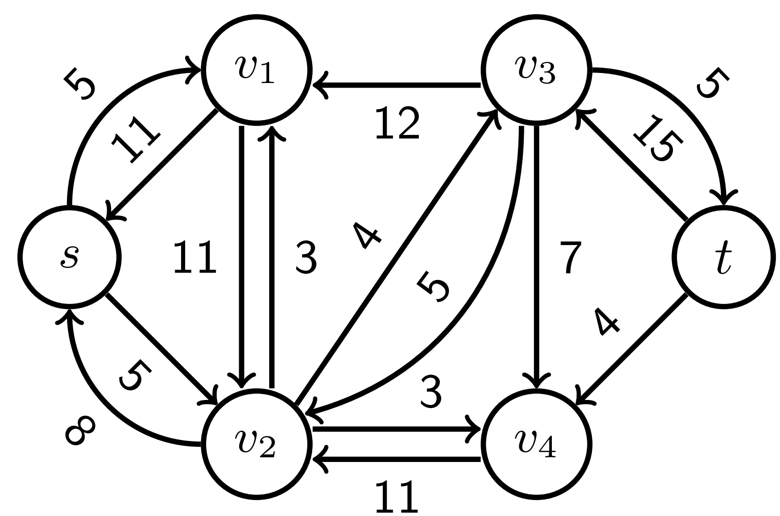

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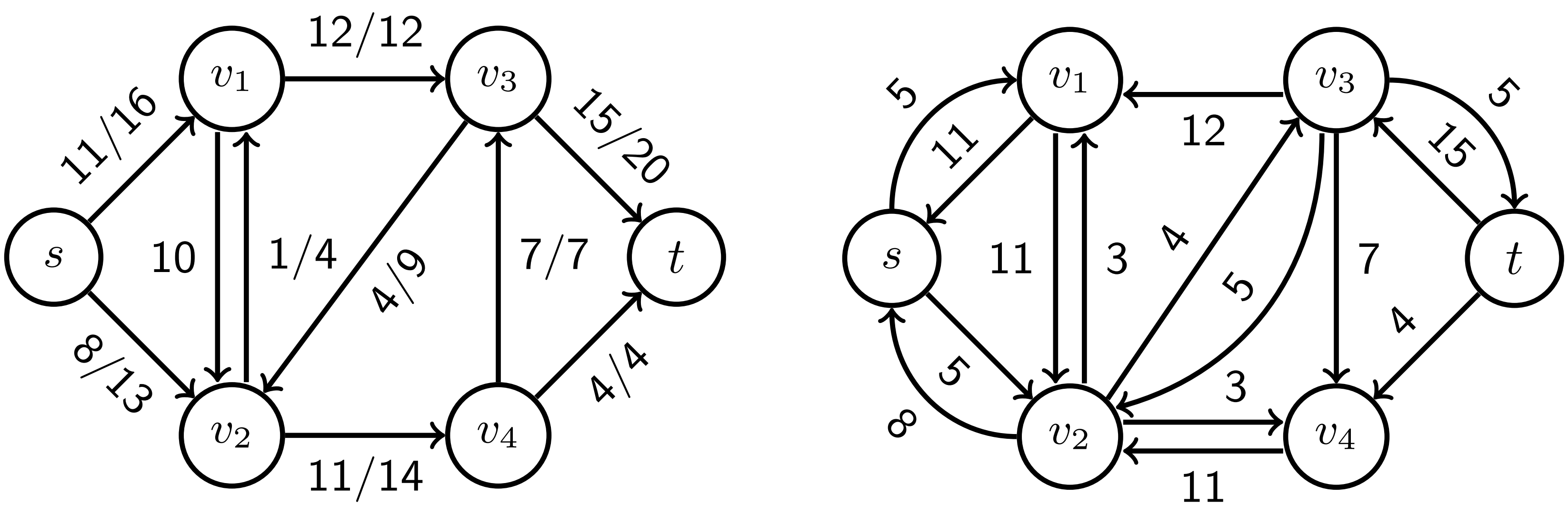

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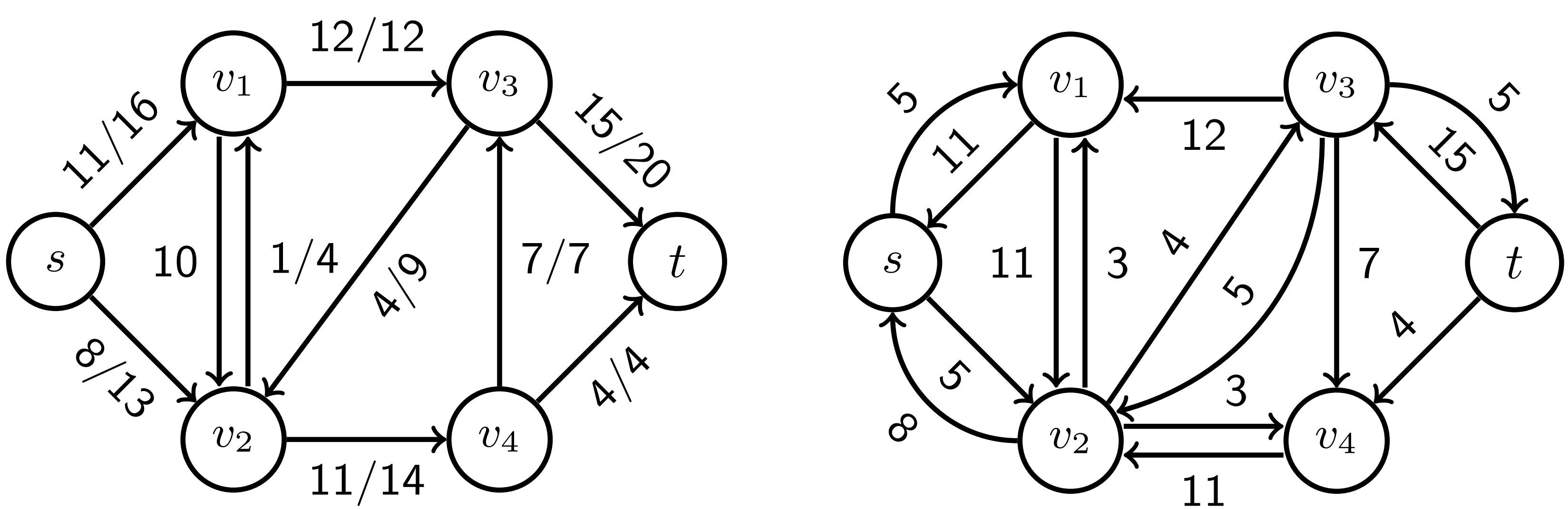

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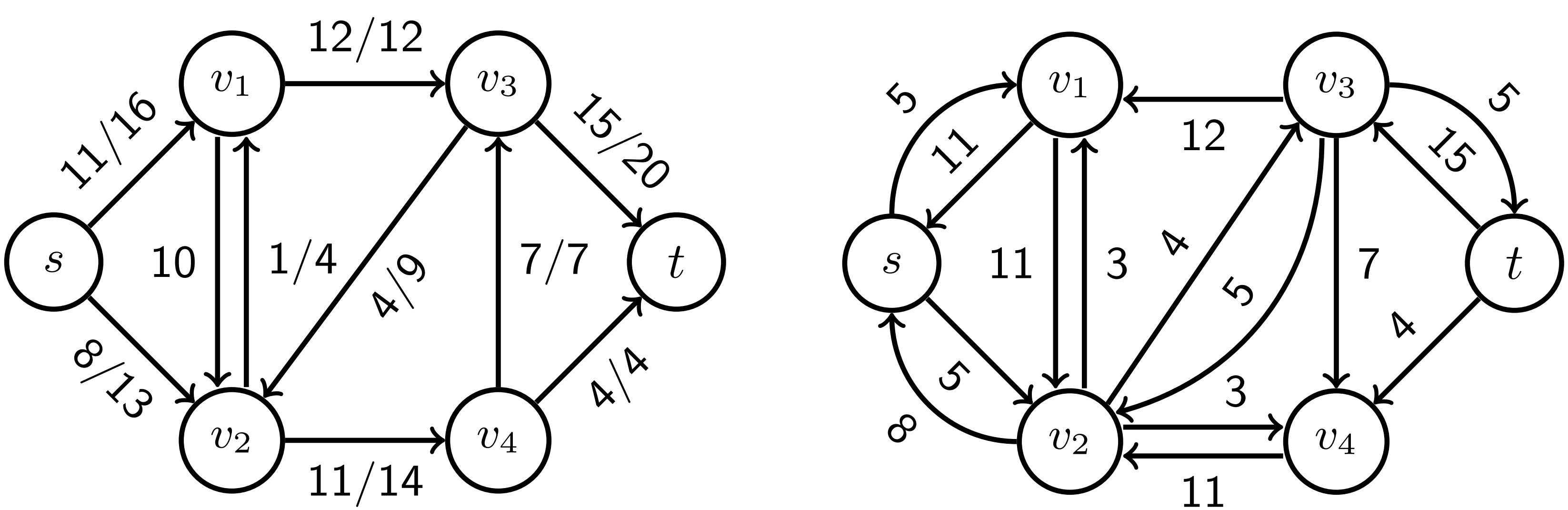

A path p from s to t in G_f is called an *augmenting path*.

Max-flow–min-cut theorem

Let f be a *flow* in a *flow network* $G = (\mathcal{V}, \mathcal{E}, c, s, t)$. Then the following conditions are equivalent:

1) f is a maximal flow in G
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Hence $|f| = \text{cut}(\mathcal{S}, \mathcal{T})$ is maximal (equivalently $\text{cut}(\mathcal{S}, \mathcal{T})$ is minimal)

Boykov–Kolmogorov algorithm

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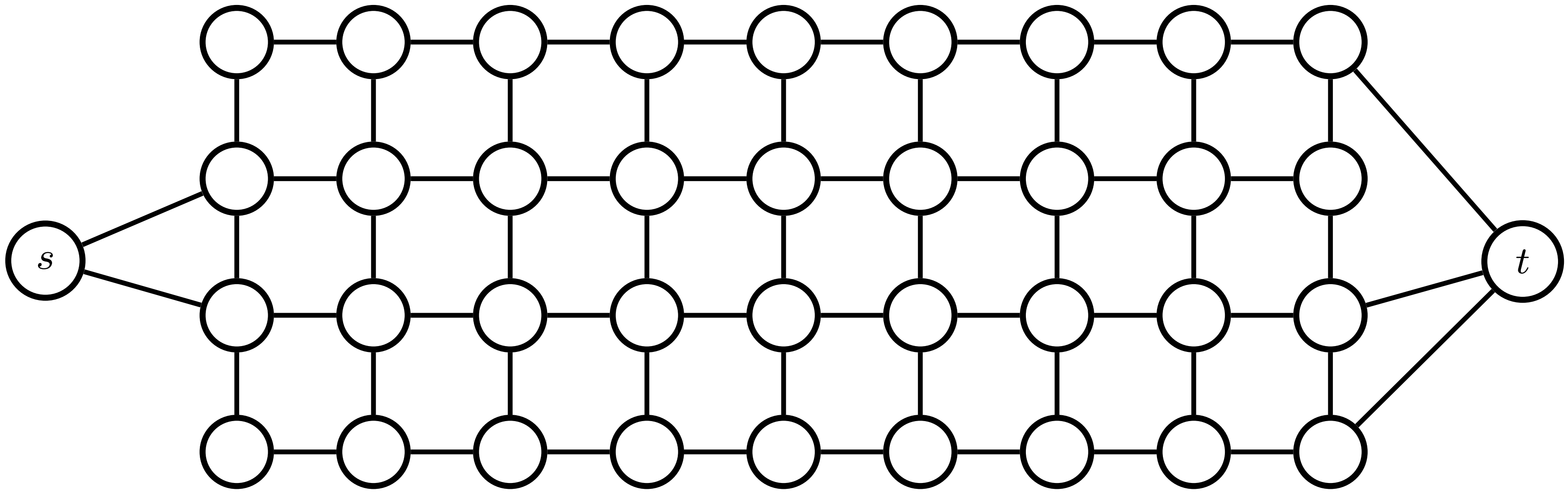

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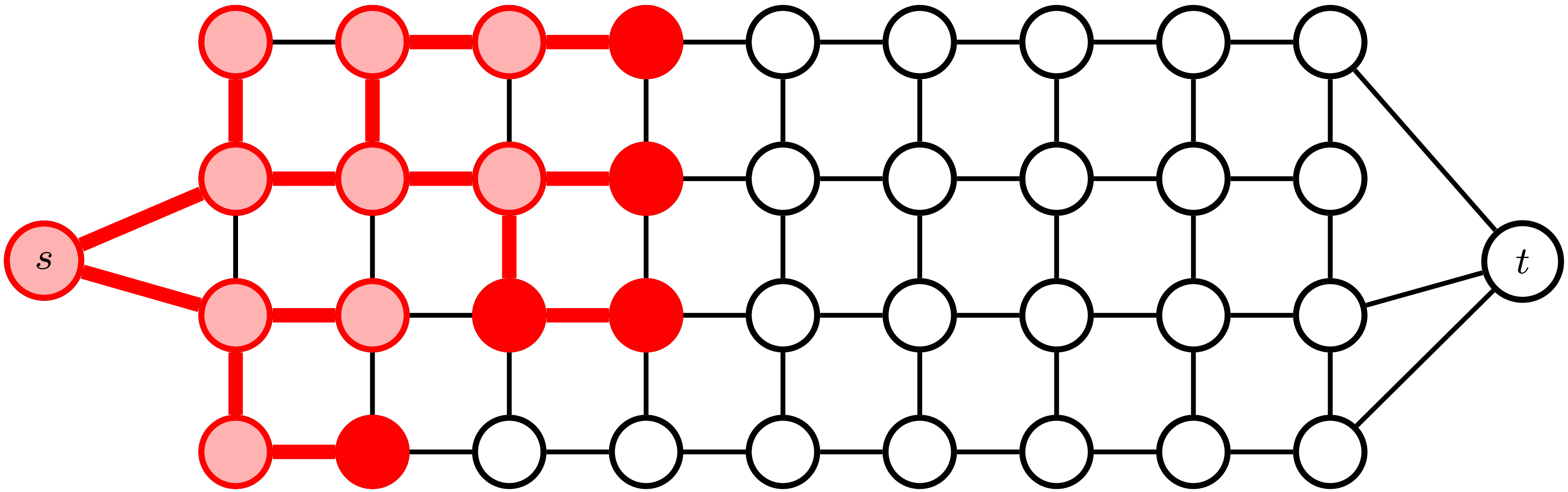

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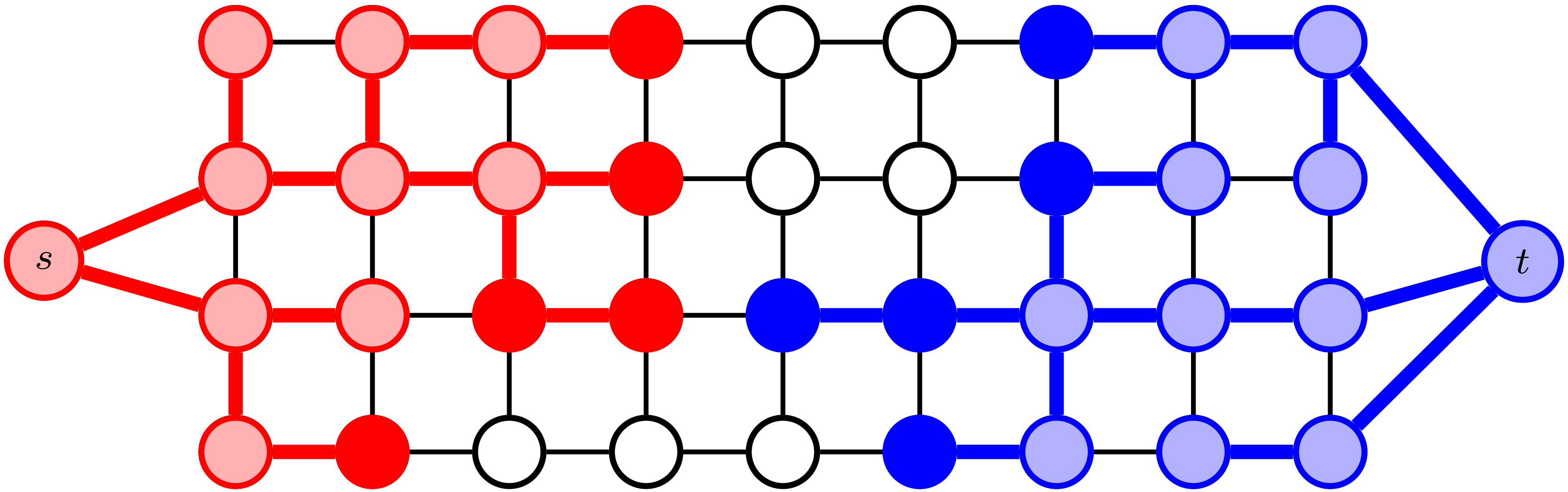

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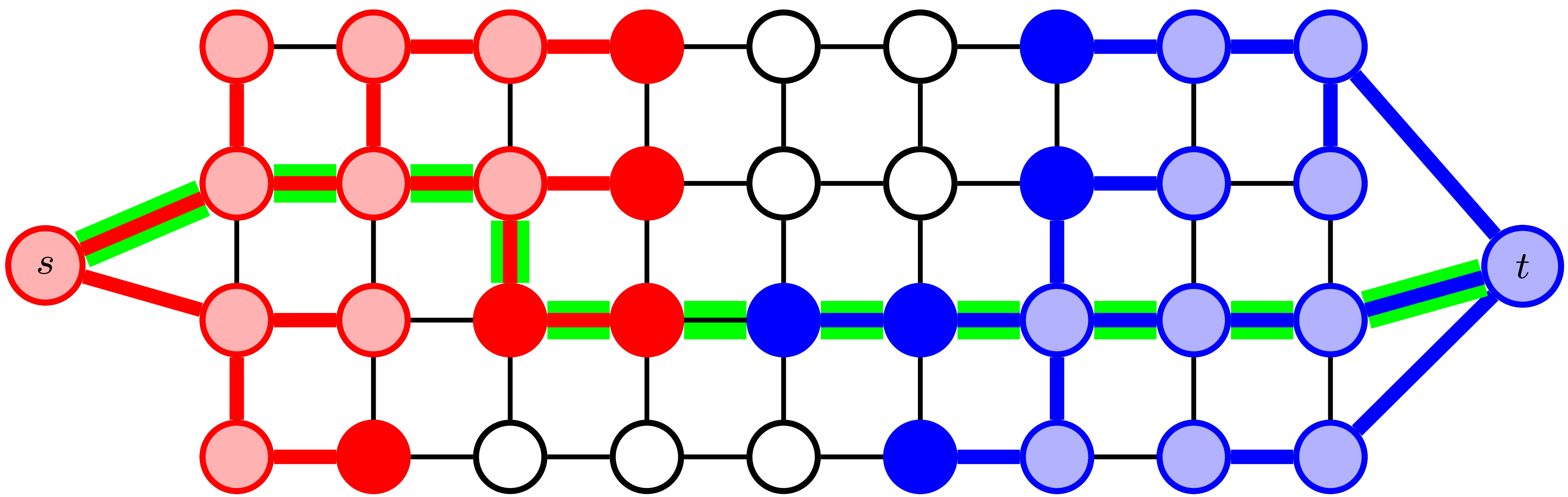

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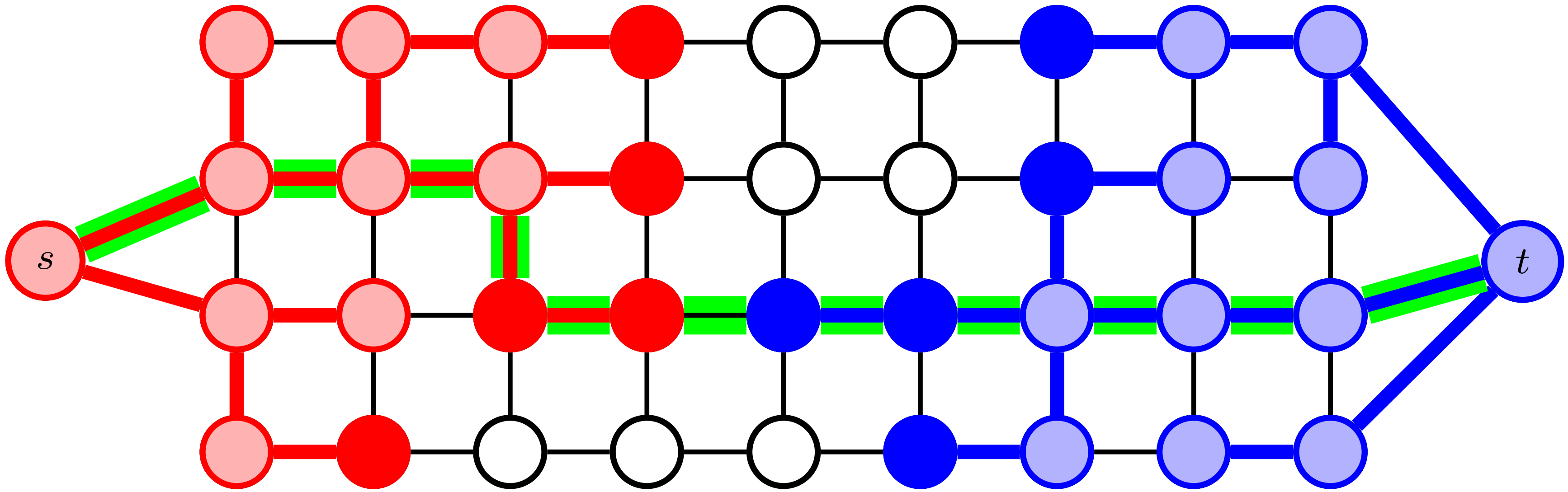
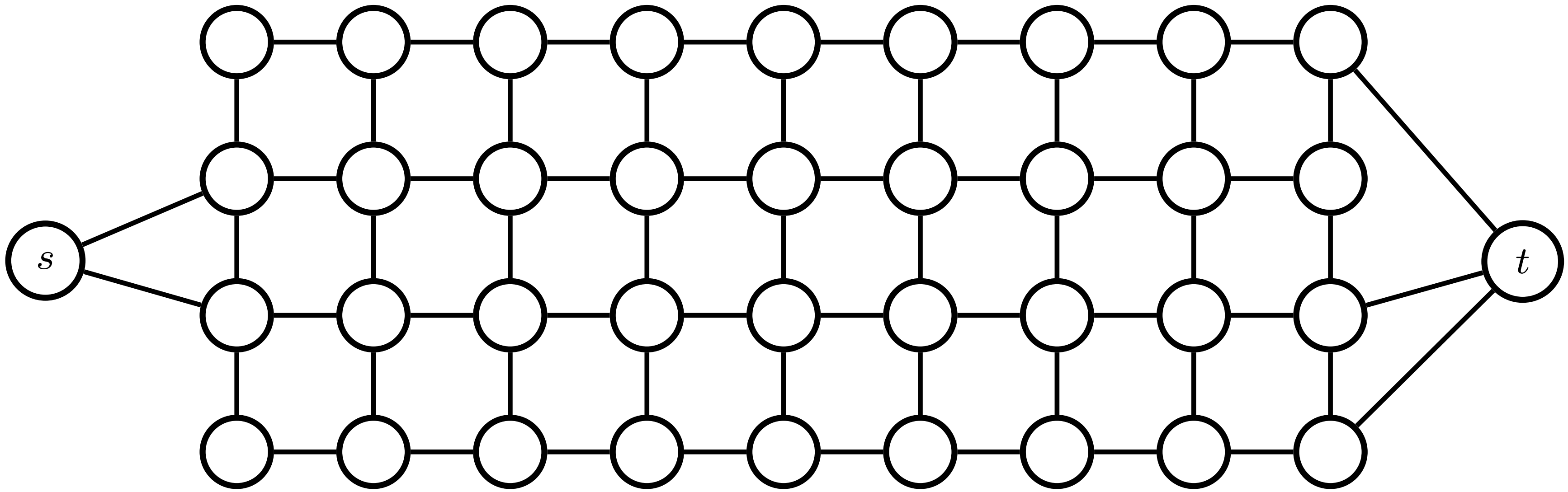
- Active nodes:
- Passive nodes:
- Free nodes:

Boykov–Kolmogorov algorithm




```
while true do
    grow S or T to find an augmenting path P from s to t
    if P = ∅ then
        terminate
    end if
    augment on P
    adopt orphans
end while
```

Growth stage



- The *active nodes* explore adjacent edges and acquire new children from a set of *free nodes*
- The newly acquired nodes become *active* members of the corresponding search trees
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- If an *active node* encounters a neighboring node belonging to the opposite tree, the growth stage terminates

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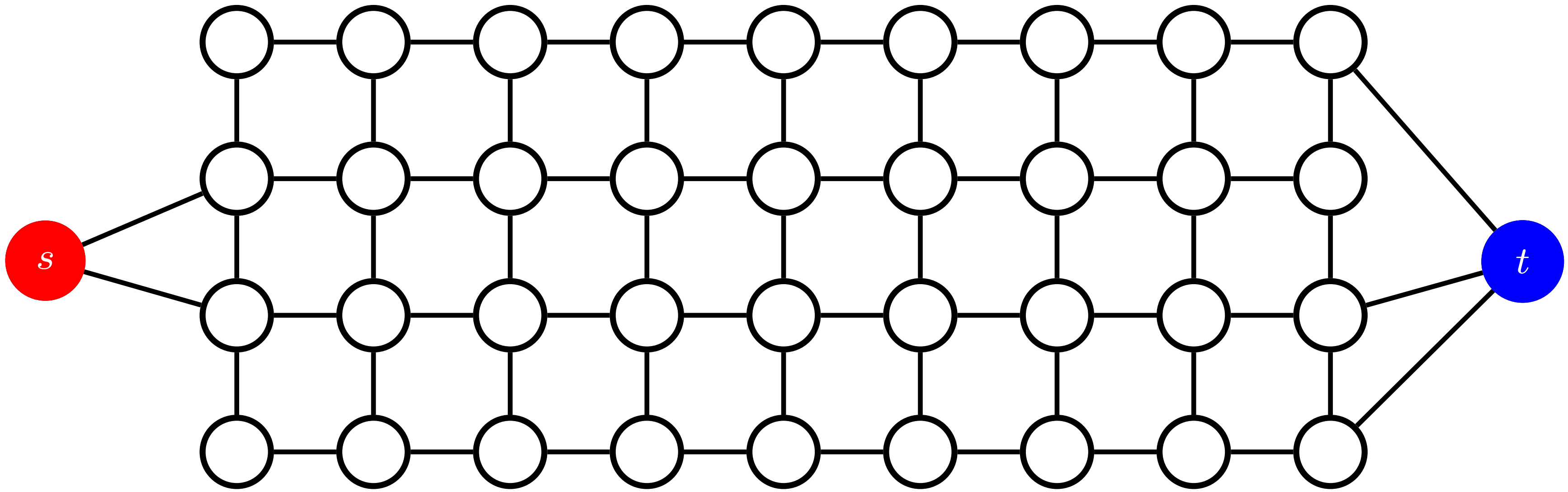

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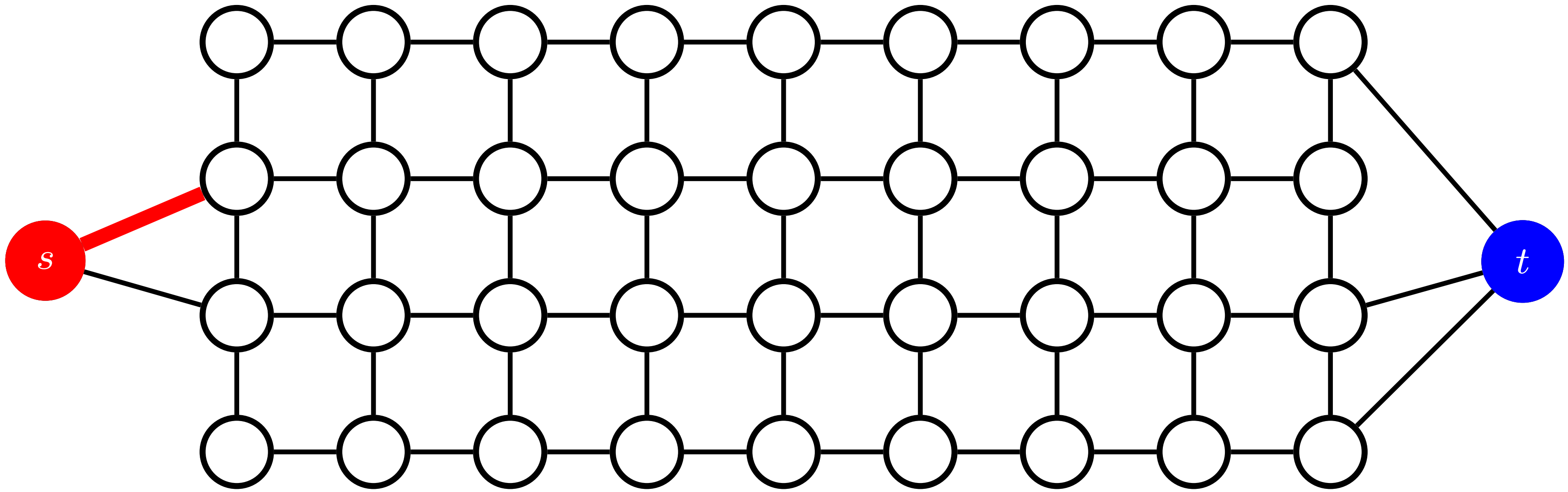

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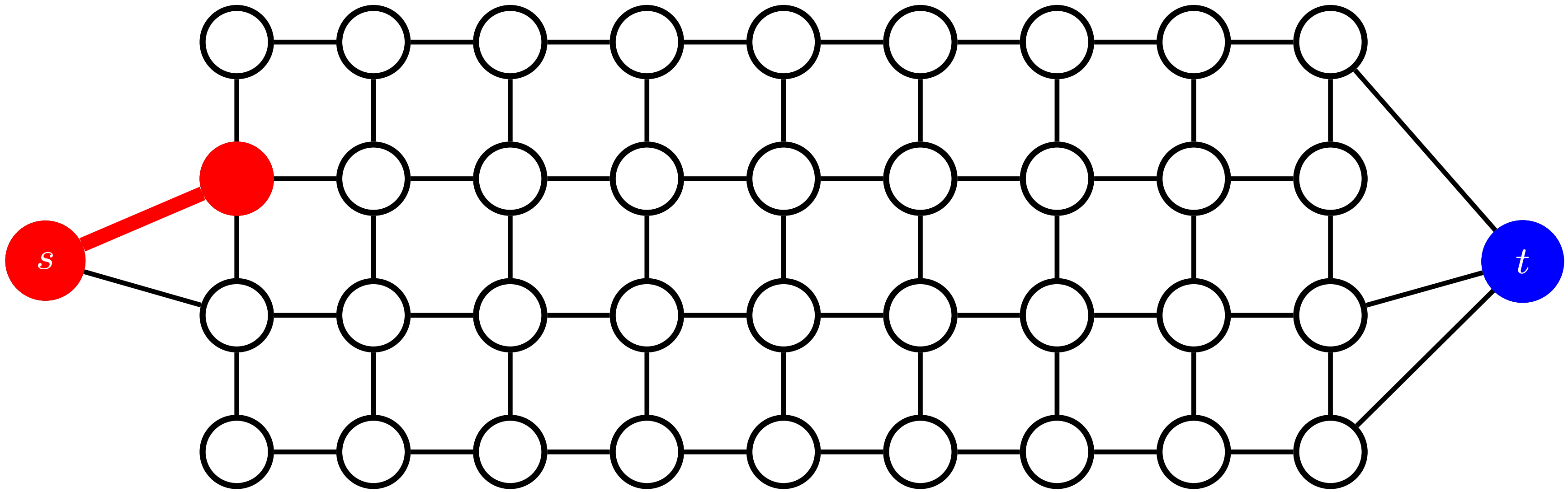

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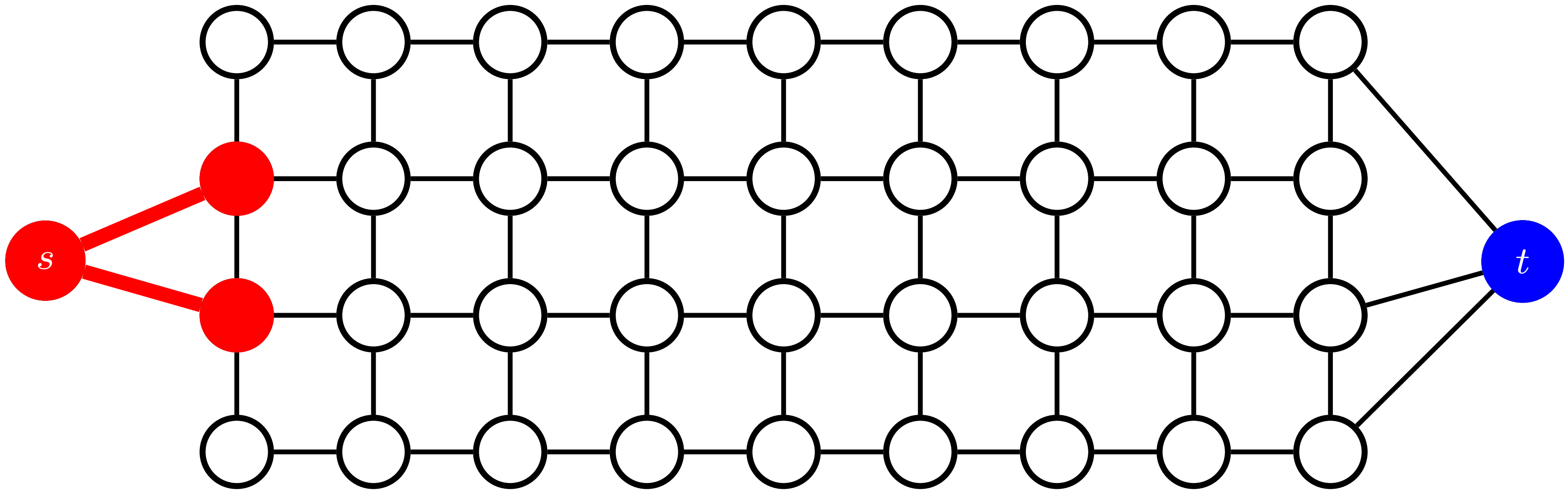

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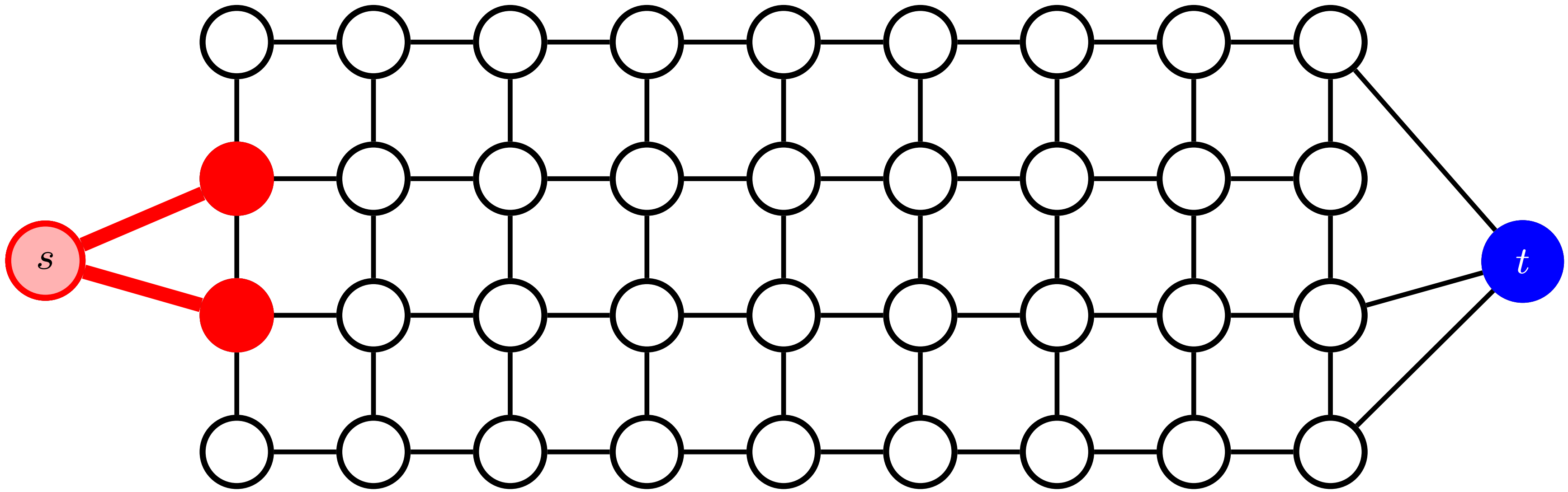

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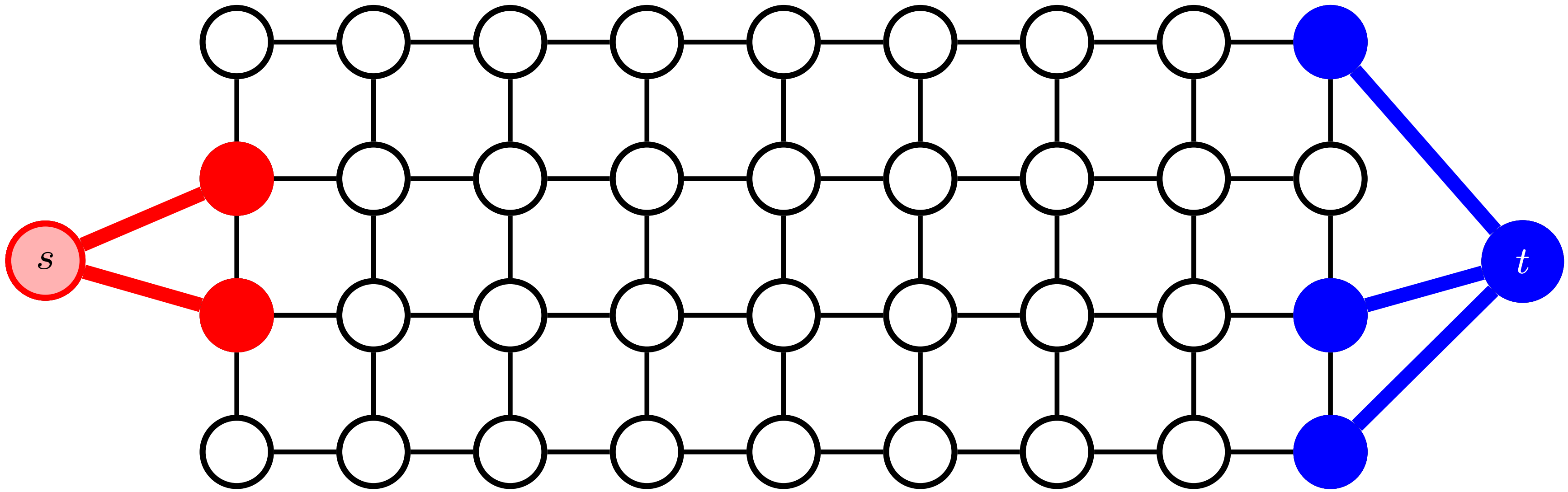

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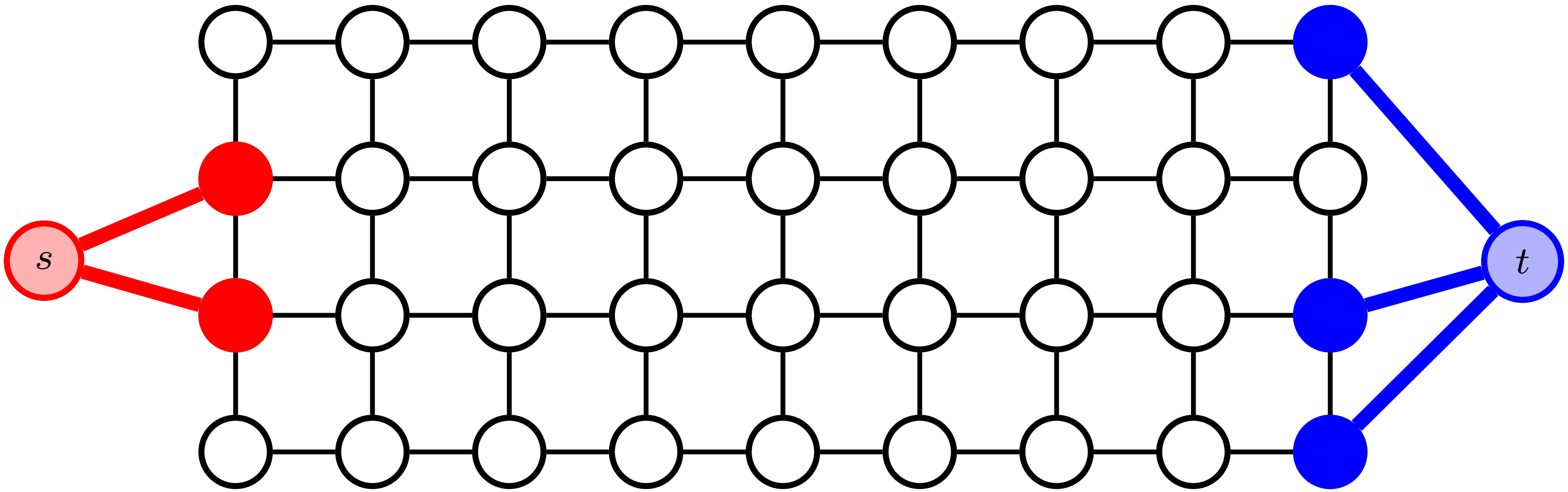

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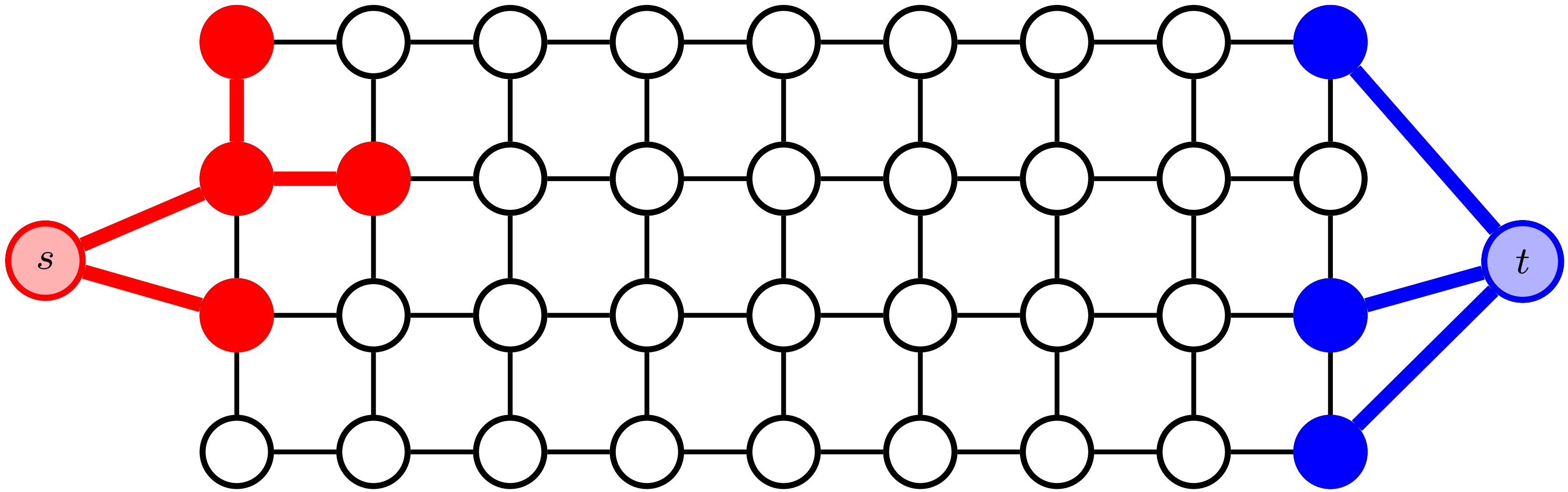

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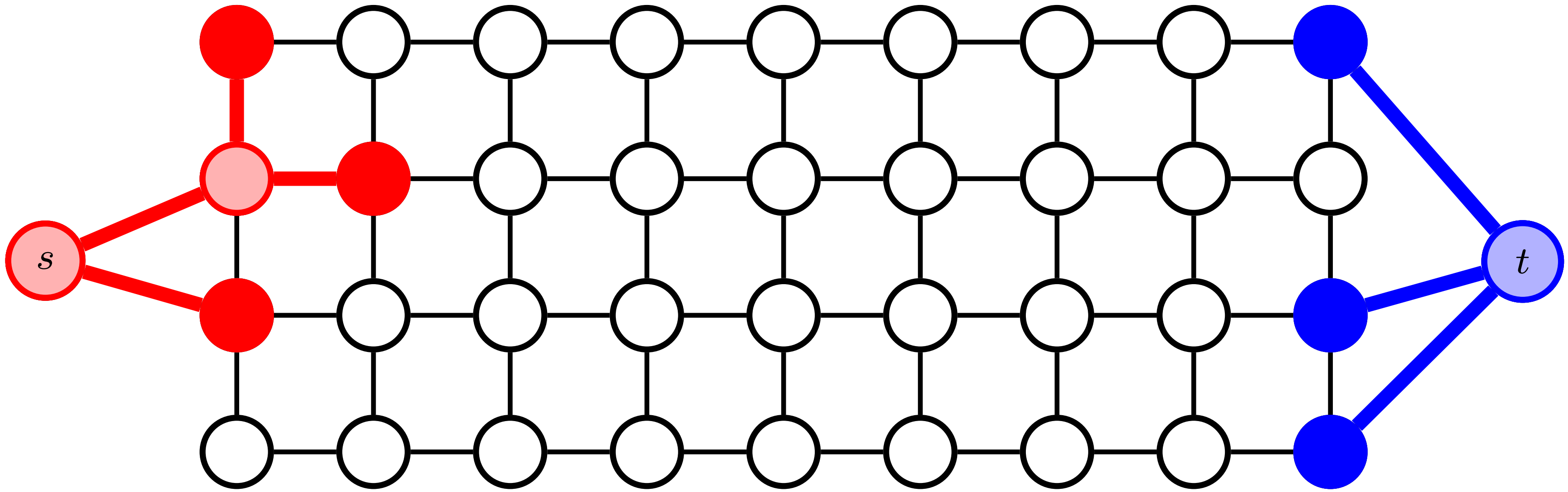

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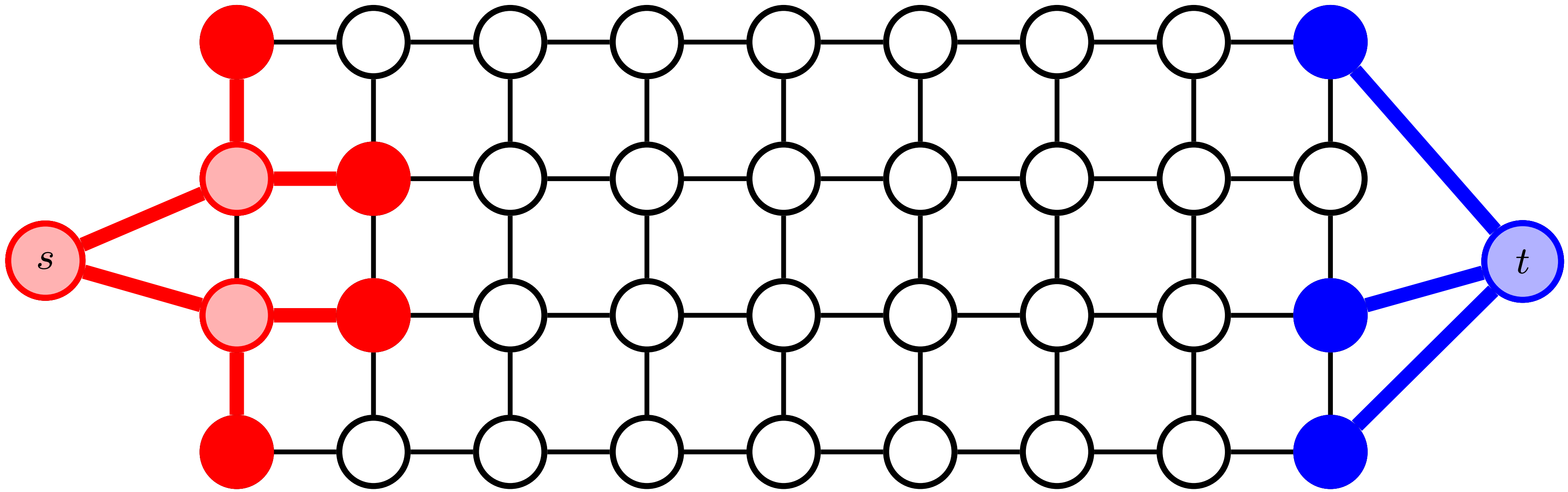

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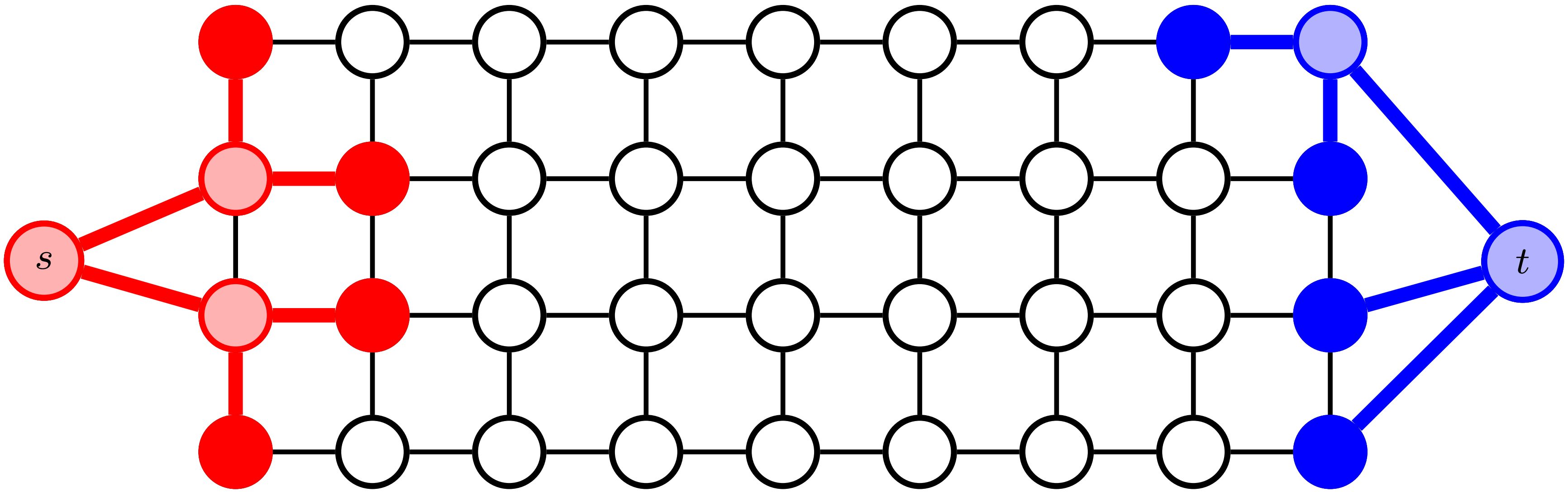

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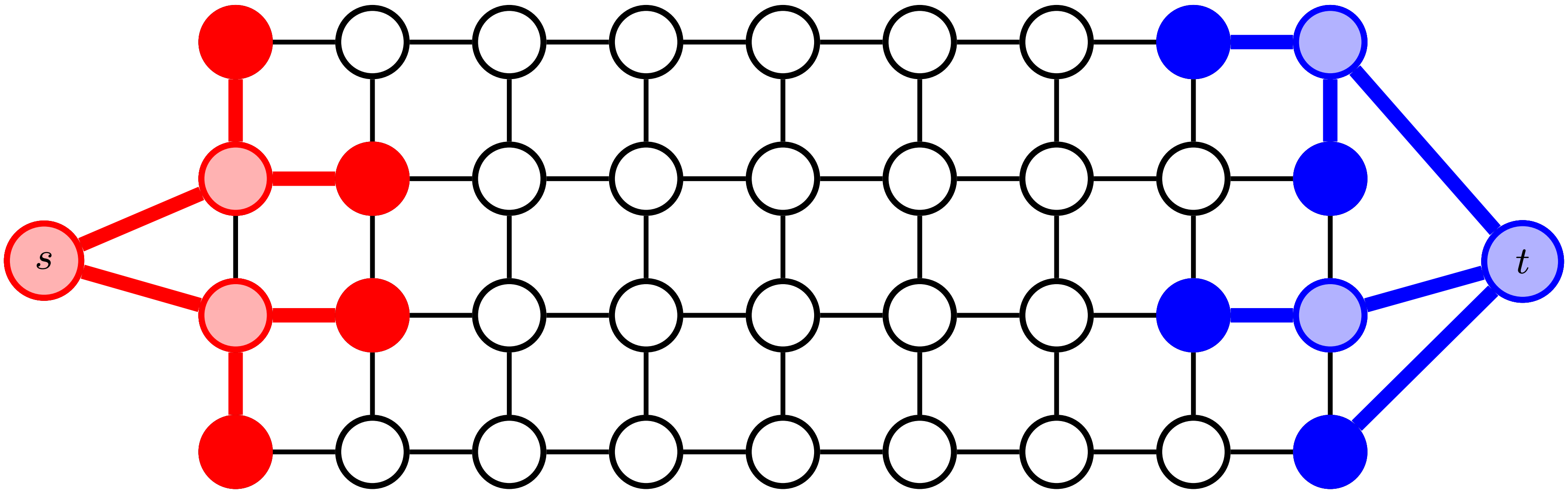

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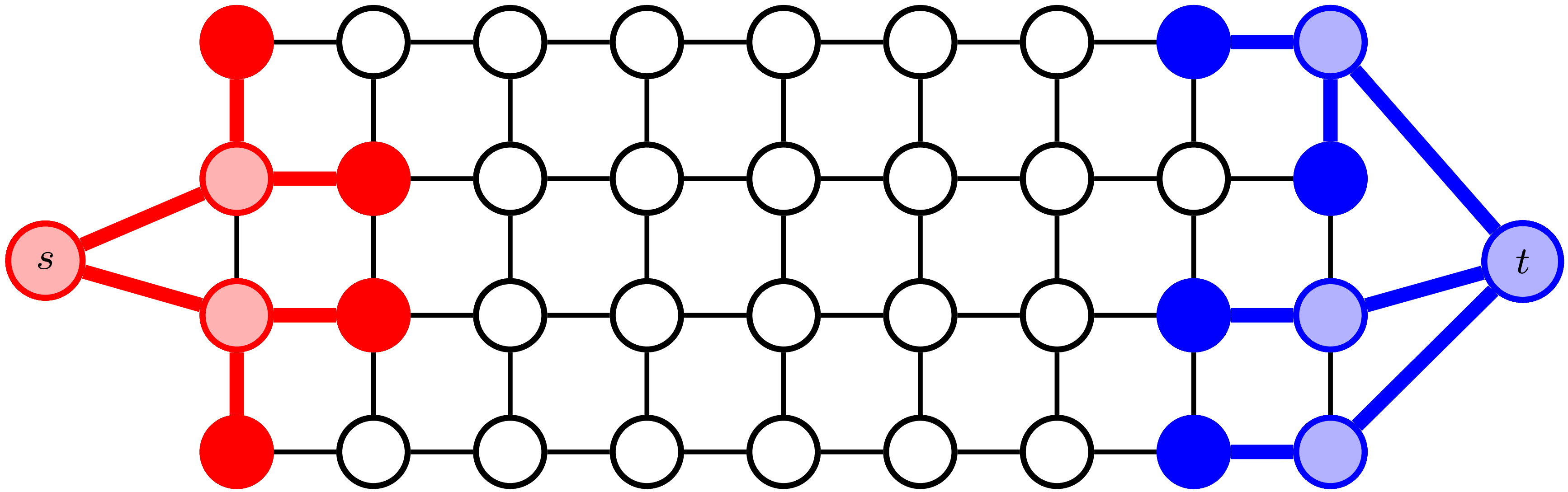

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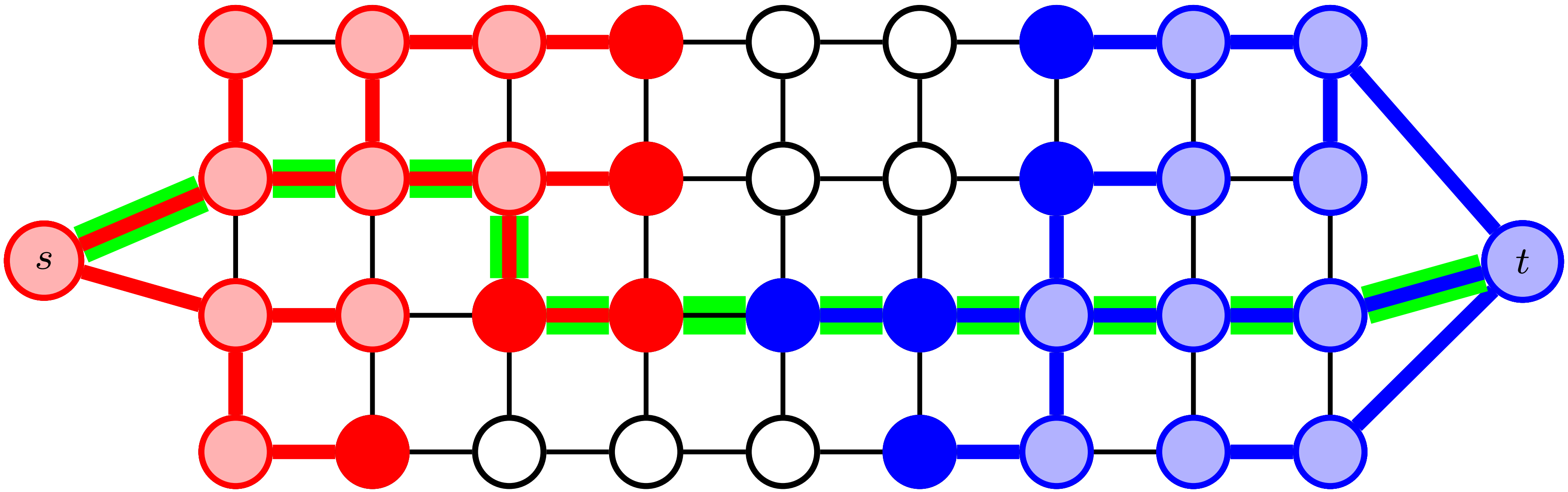

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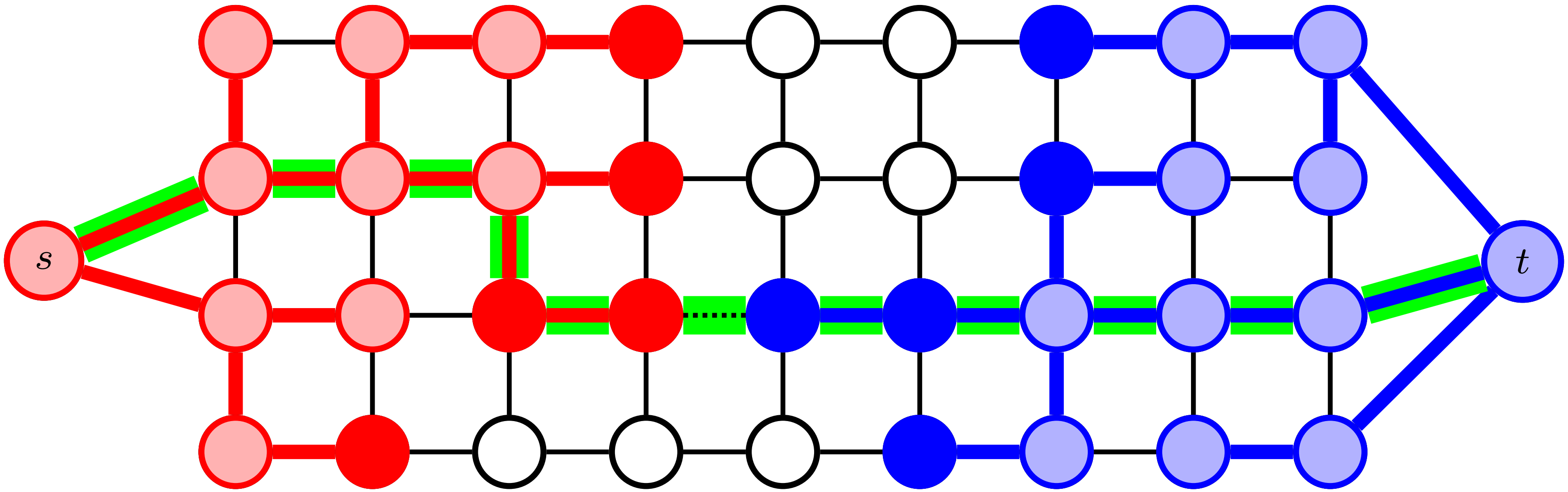

- Find the bottleneck capacity Δ on P
- Update the residual graph by pushing flow Δ through P

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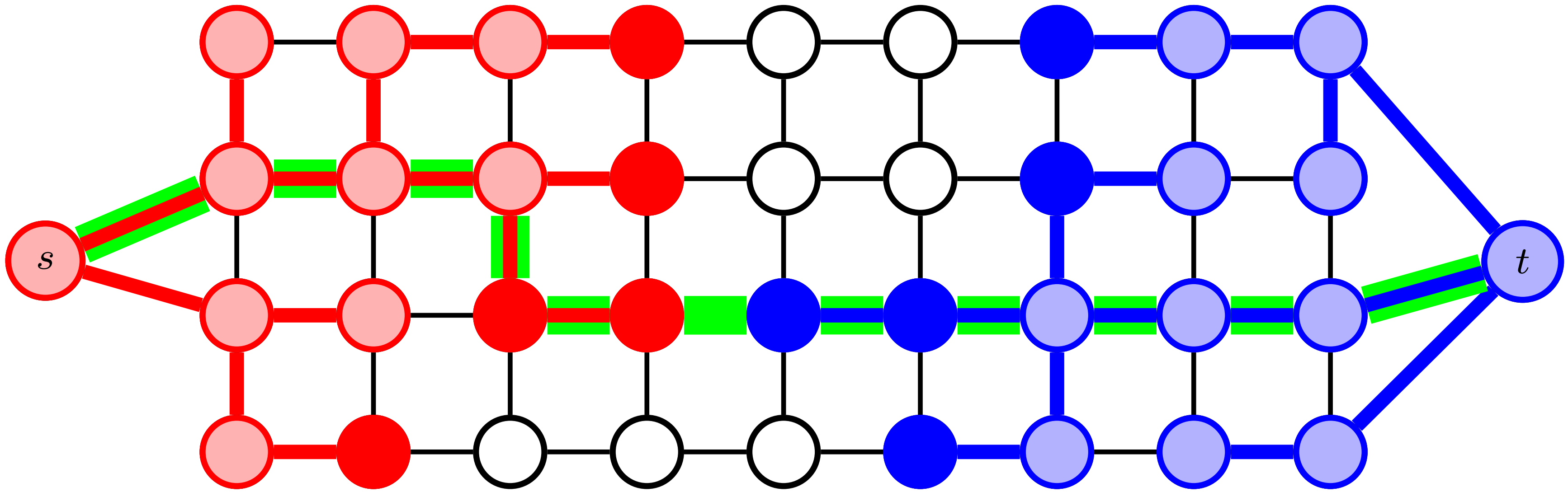

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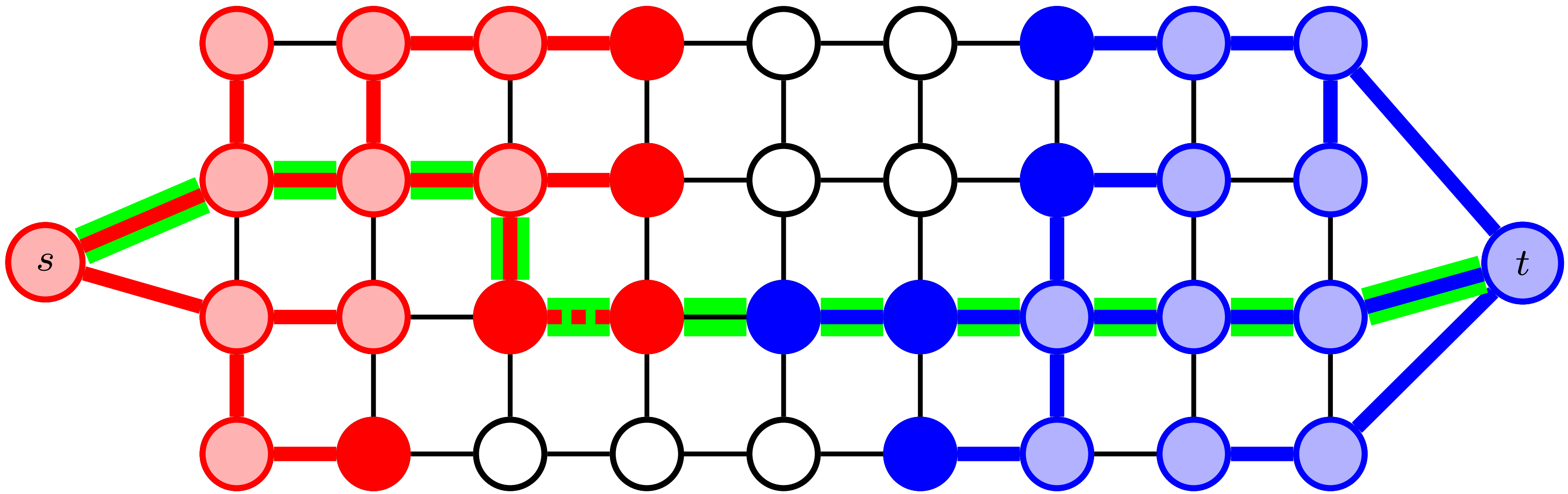
Adoption stage

- *Orphan* (○ ○): the nodes such that the edges linking them to their parents are no longer valid (i.e. they are saturated)
- By removing them the search trees S and T may be split into *forests*



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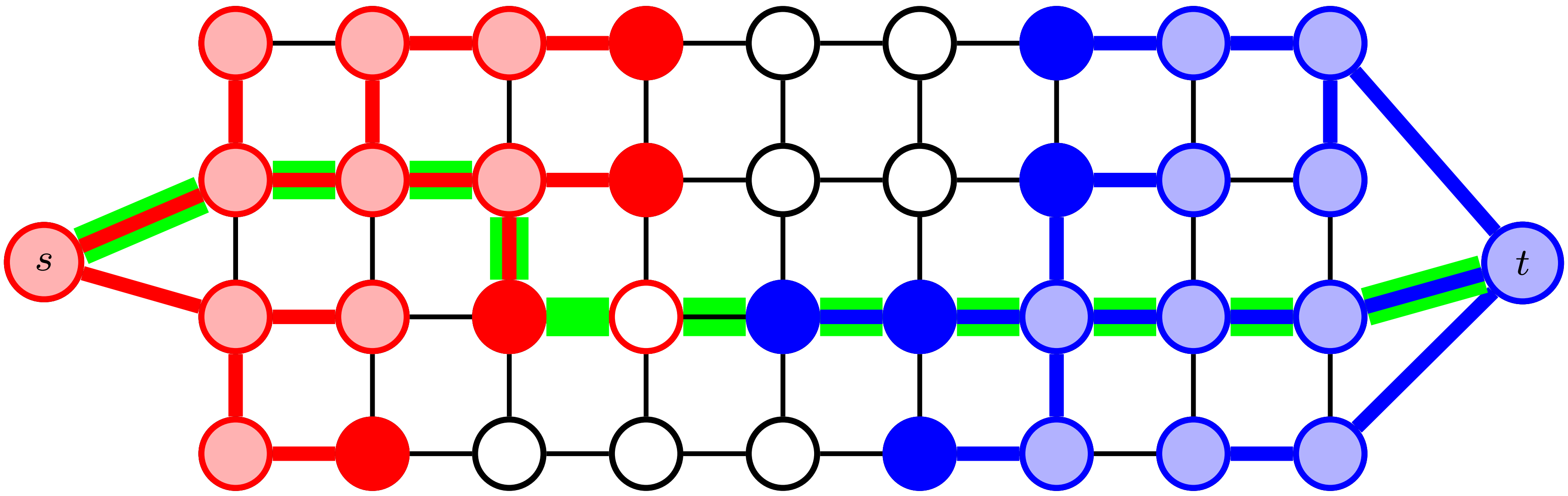

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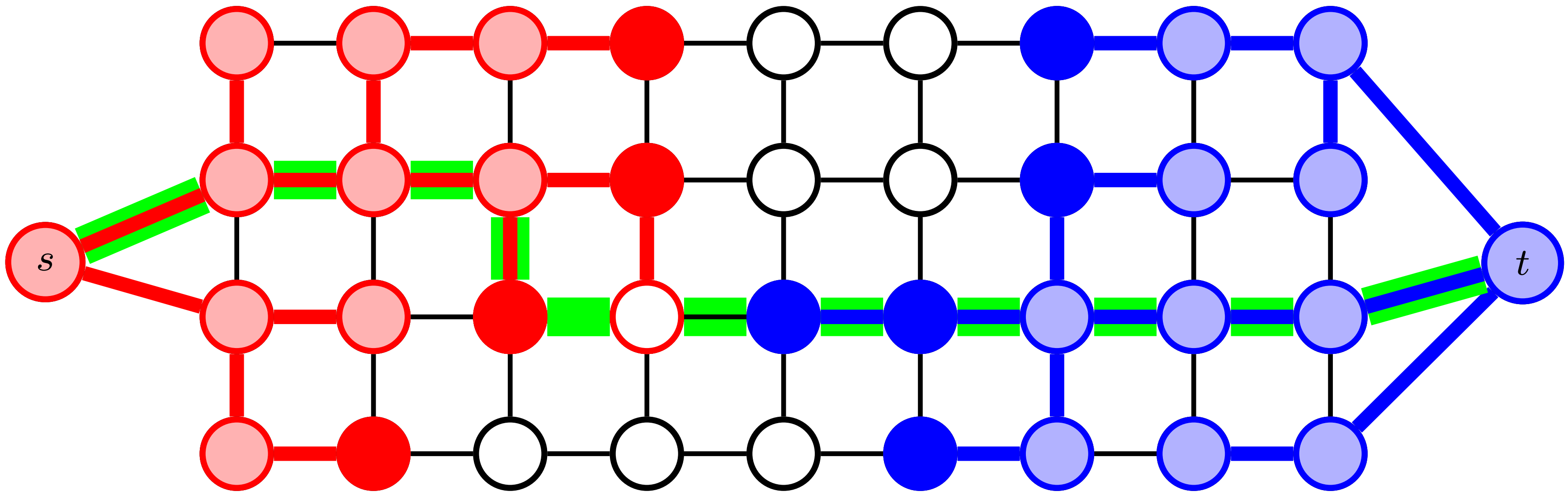

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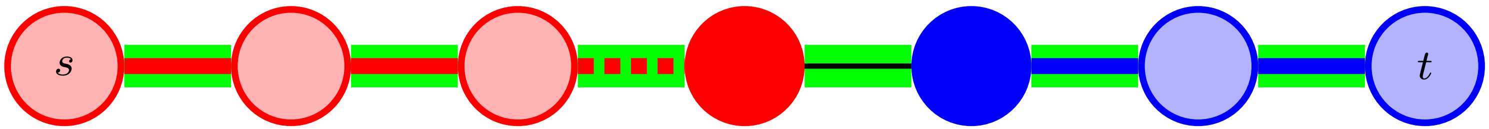
If an orphan p does not find a valid parent then it becomes a *free node*



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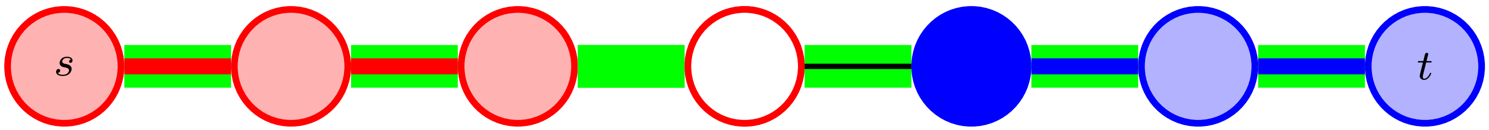

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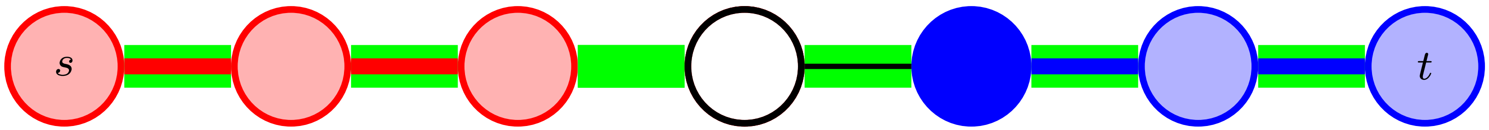

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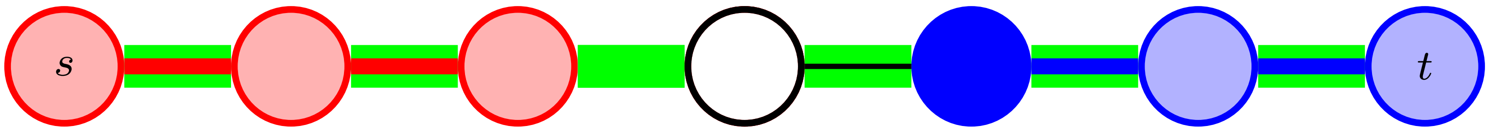
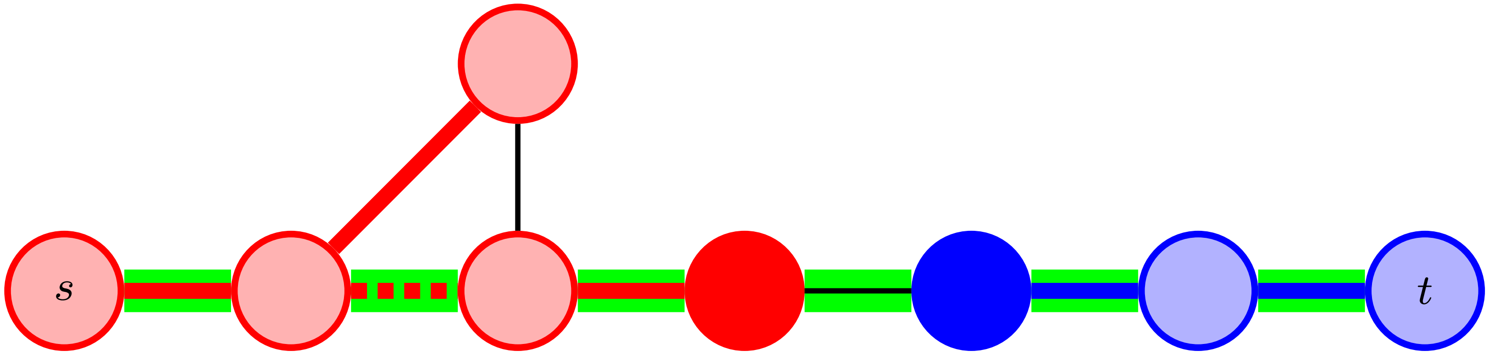


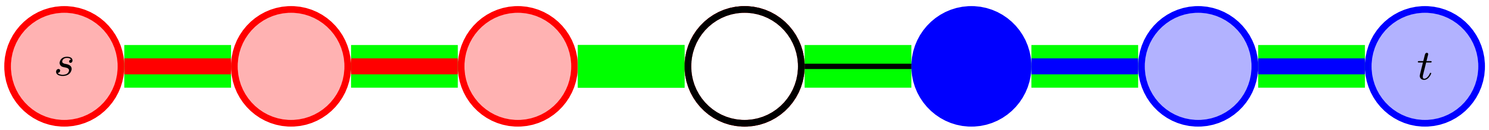
Scan all neighbors q of p such that q belong to the same tree as p:

- if tree $c(q, p) > 0$, add q to the *active set*
- if $\texttt{parent}(q) = p$, add q to the set of *orphans* and set $\texttt{parent}(q) = \varnothing$



Adoption stage

If an orphan p does not find a valid parent then it becomes a *free node*



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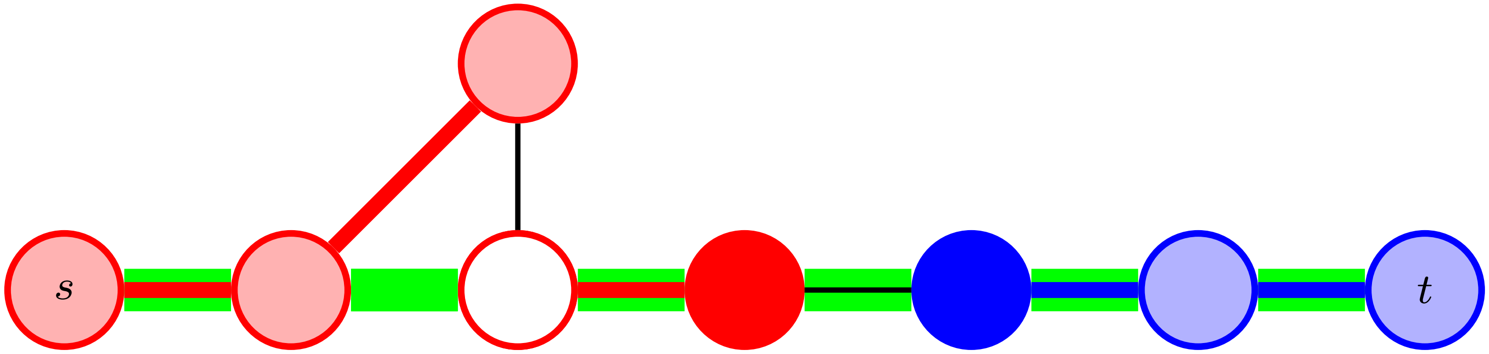

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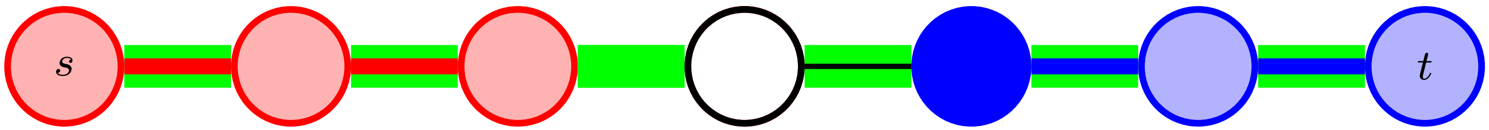

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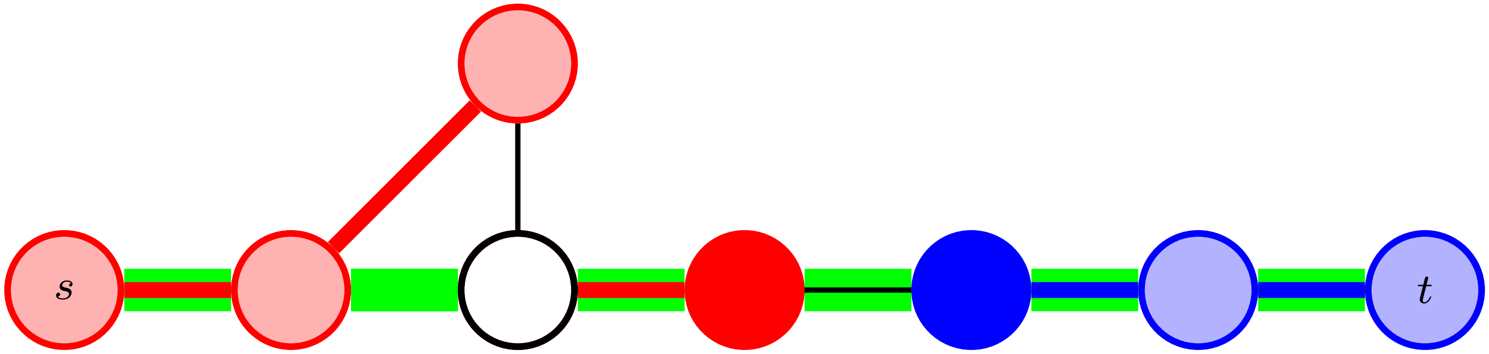

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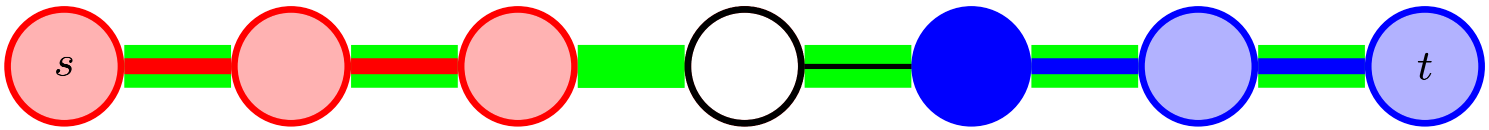

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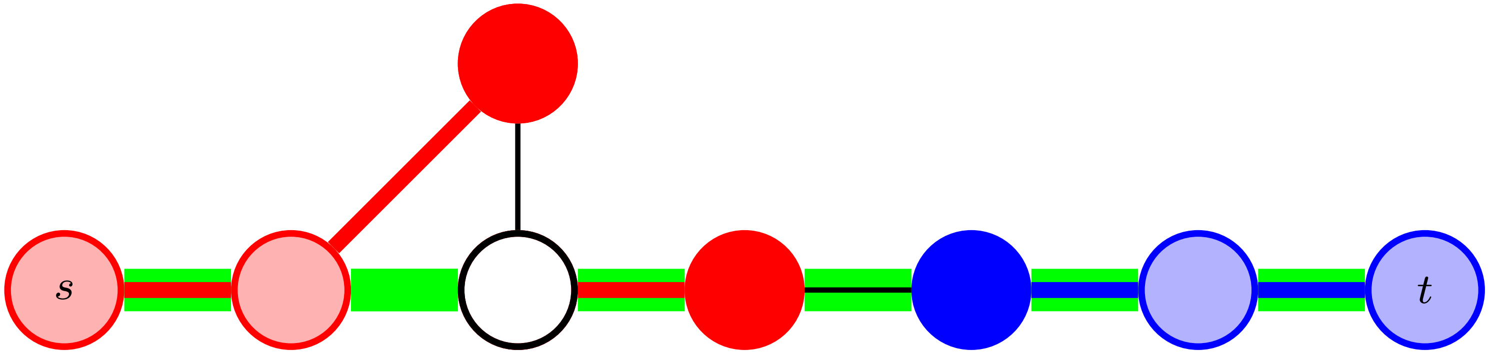

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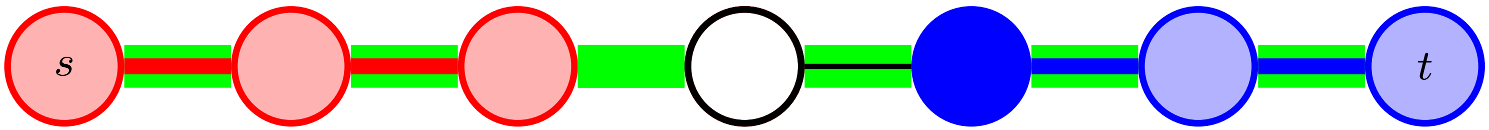

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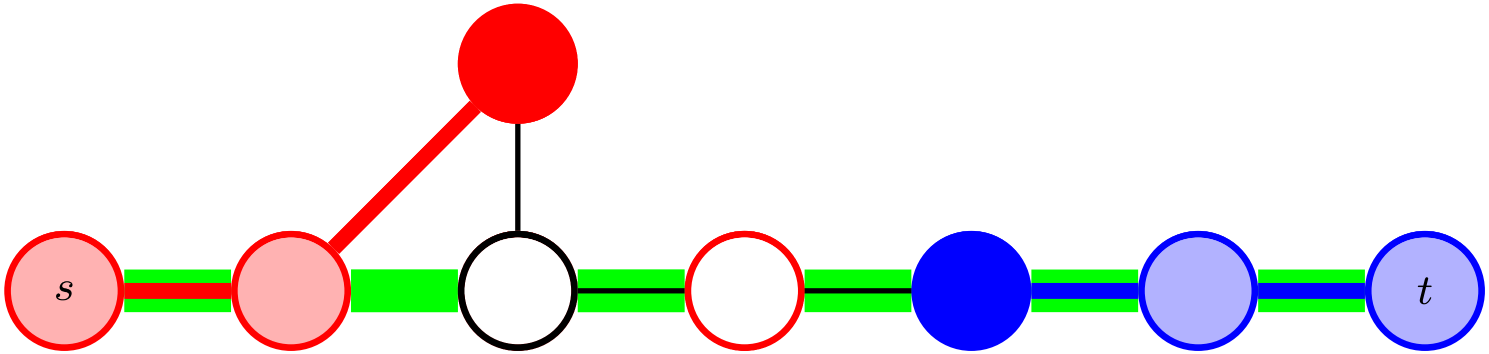

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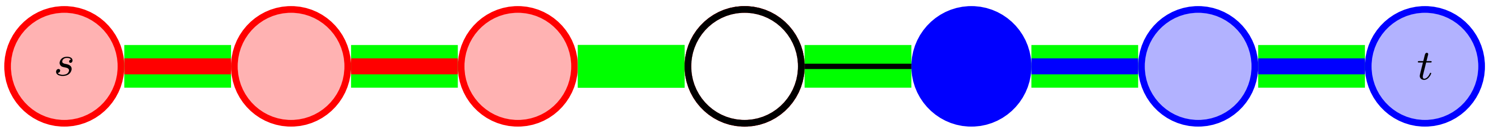

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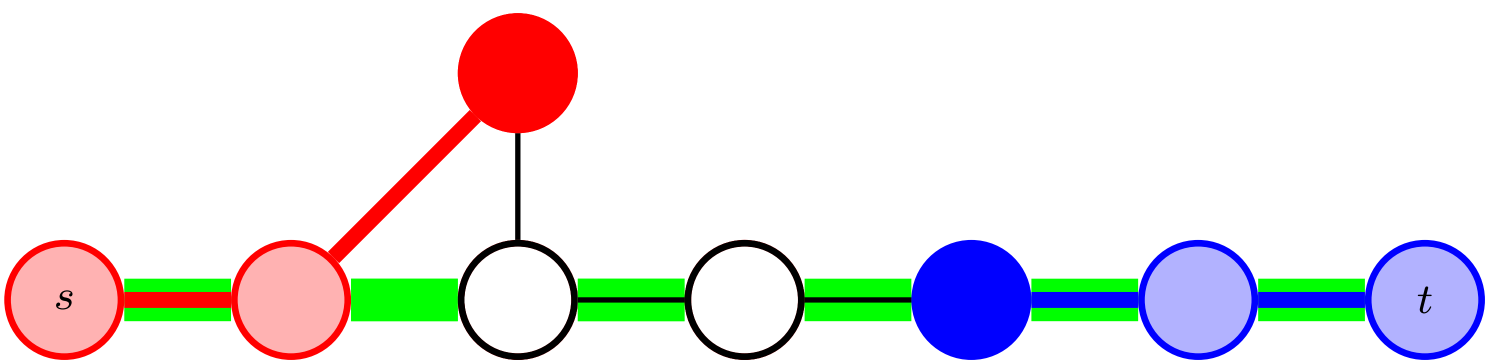

Summary

- `Boykov-Kolmogorov algorithm` is an *augmented path-based method* with worst case complexity $\mathcal{O}(|\mathcal{E}| \cdot |\mathcal{V}|^2 \cdot |C|)$, where $|C|$ is the value of the minimum cut
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- In many computer vision problems we aim to minimize an em energy function

$$E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j \in \mathcal{E})} E_{ij}(y_i, y_j)$$

- As we will see, this is often achieved by solving the *maxFlow problem*