Seminar for Image Segmentation and Shape Analysis
(IN2107)

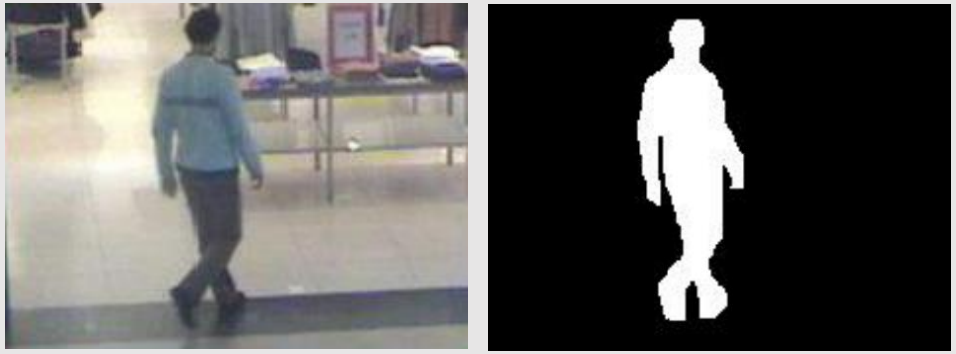
Wintersemester 2016/2017

What energy functions can be minimized via graph cuts?

Name : Jude Ng
Date : 26.10.2016

1. Motivation

- Many vision problems can be expressed in terms of energy minimization
- The goal is to find a labeling $f : P \to L$
- Example: foreground extraction



Original image and the extracted foreground *

A standard form of an energy function

$$E(f) = E_{data} + E_{smoothness}$$

$$E_{smoothness}(f) = \sum_{\{p,q\} \in N} V_{p,q}(f_p, f_q)$$

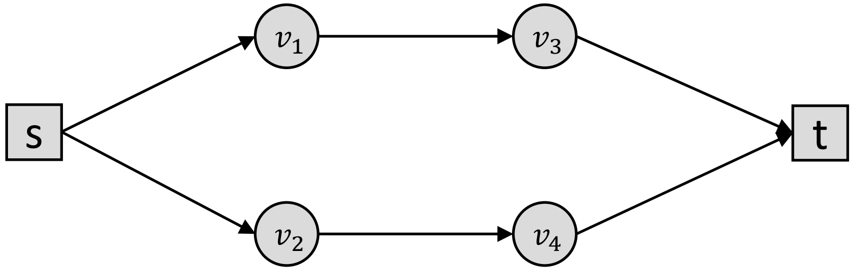
$$E_{data}(f) = \sum_{p \in P} D_p(f_p)$$

- E is non-convex with a high-dimensional space and is difficult to minimize
- Usually solved with simulated annealing, which is slow in practice

* Images from: *S. Denman, C. Fookes and S. Sridharan, "Improved Simultaneous Computation of Motion Detection and Optical Flow for Object Tracking", DICTA '09*

1. Motivation

- A recent approach to minimize E based on graph cuts
- Basic technique: construct a specialized graph to represent E



Graph representability

$\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{\mathcal{E}})$ with terminals $\boldsymbol{s}$ and $\boldsymbol{t}$
and a subset $\boldsymbol{V_o} = \{\boldsymbol{v_1}, \dots, \boldsymbol{v_n}\} \subset \boldsymbol{V} - \{\boldsymbol{s}, \boldsymbol{t}\}$

A cut $\boldsymbol{C}$ partitions the $\boldsymbol{V_o}$ into two sets $\boldsymbol{S}$ and $\boldsymbol{T}$,
where $\boldsymbol{v_i} \in \boldsymbol{S}$ if $\boldsymbol{x_i} = \boldsymbol{0}$ and $\boldsymbol{v_i} \in \boldsymbol{T}$ if $\boldsymbol{x_i} = \boldsymbol{1}$

$\boldsymbol{E}(\boldsymbol{x_1}, \dots, \boldsymbol{x_n})$ is equal to the minimum s-t cut among all cuts $\boldsymbol{C}$

- The minimum cut on the graph minimizes E
- Minimum cut can be computed efficiently with the Ford-Fulkerson max-flow algorithm
- Problem: Graph construction is complex
- Is there a class of E that can be minimized via graph cuts?

2. The class $\mathcal{F}^2$

- Functions that can be written as a **sum of functions of up to two variables**

$$E(x_1, \dots, x_n) = \sum_i E^i(x_i) + \sum_{i<j} E^{i,j}(x_i, x_j)$$

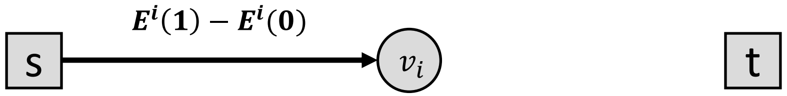
- E is graph-representable if and only if each term $E^{i,j}$ satisfies the **regularity** condition

$$E^{i,j}(0,0) + E^{i,j}(1,1) \leq E^{i,j}(0,1) + E^{i,j}(1,0)$$

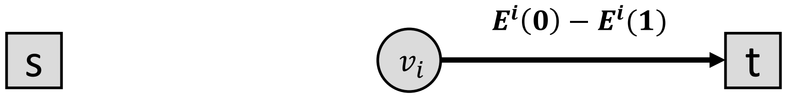
- Regularity is analogous to **submodular functions**
- Non-regular functions are **NP-hard** to minimize

2. The class $\mathcal{F}^2$

- A term E^i depending on **one variable** x_i
- Case 1: $E^i(0) < E^i(1)$



- Case 2: $E^i(1) < E^i(0)$



2. The class $\mathcal{F}^2$

- A term $E^{i,j}$ depending on **two variables** x_i and x_j can be rewritten as follows:

$$\boldsymbol{E^{i,j}} = \begin{array}{|c|c|}\hline \boldsymbol{E^{i,j}(0,0)} & \boldsymbol{E^{i,j}(0,1)} \\ \hline \boldsymbol{E^{i,j}(1,0)} & \boldsymbol{E^{i,j}(1,1)} \\ \hline \end{array} = \begin{array}{|c|c|}\hline \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \\ \hline \end{array}$$

2. The class $\mathcal{F}^2$

- Expansion of $E^{i,j}$

$$\begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline \end{array} = A + \begin{array}{|c|c|}\hline 0 & B-A \\ \hline C-A & D-A \\ \hline \end{array}$$

$$= A + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline C-A & C-A \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & B-A \\ \hline 0 & D-A-(C-A) \\ \hline \end{array}$$

$$= A + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline C-A & C-A \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & B-A \\ \hline 0 & D-C \\ \hline \end{array}$$

2. The class $\mathcal{F}^2$

- Expansion of $E^{i,j}$

$$\begin{array}{|c|c|}\hline \mathbf{0} & \boldsymbol{B-A} \\ \hline \mathbf{0} & \boldsymbol{D-C} \\ \hline \end{array} = \begin{array}{|c|c|}\hline \mathbf{0} & \boldsymbol{D-C} \\ \hline \mathbf{0} & \boldsymbol{D-C} \\ \hline \end{array} + \begin{array}{|c|c|}\hline \mathbf{0} & \boldsymbol{B-A} \\ & \boldsymbol{-(D-C)} \\ \hline \mathbf{0} & \mathbf{0} \\ \hline \end{array}$$

$$= \begin{array}{|c|c|}\hline \mathbf{0} & \boldsymbol{D-C} \\ \hline \mathbf{0} & \boldsymbol{D-C} \\ \hline \end{array} + \begin{array}{|c|c|}\hline \mathbf{0} & \boldsymbol{B+C-A-D} \\ \hline \mathbf{0} & \mathbf{0} \\ \hline \end{array}$$

- Satisfying the regularity condition

$$B + C - A - D > 0$$
$$E^{i,j}(0,1) + E^{i,j}(1,0) - E^{i,j}(0,0) - E^{i,j}(1,1) > 0$$
$$\boldsymbol{E^{i,j}(0,0) + E^{i,j}(1,1) < E^{i,j}(0,1) + E^{i,j}(1,0)}$$

2. The class $\mathcal{F}^2$

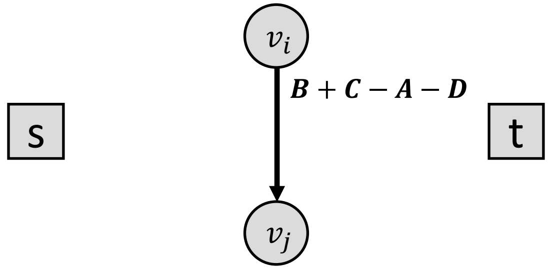
$$E^{i,j} = A + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline C-A & C-A \\ \hline\end{array} + \begin{array}{|c|c|}\hline 0 & D-C \\ \hline 0 & D-C \\ \hline\end{array} + \begin{array}{|c|c|}\hline 0 & B+C-A-D \\ \hline 0 & 0 \\ \hline\end{array}$$

Constant function (A)

Function of one variable (second and third terms)

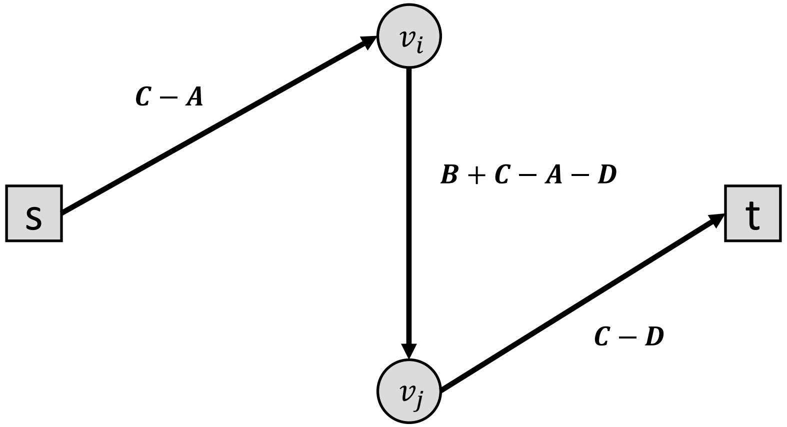
Function of two variables (fourth term)

- Adding an edge for $B + C - A - D$



2. The class $\mathcal{F}^2$

- Constructing the full graph for $E^{i,j}$
- For the case: $\boldsymbol{C - A > 0}$ **and** $\boldsymbol{D - C < 0}$



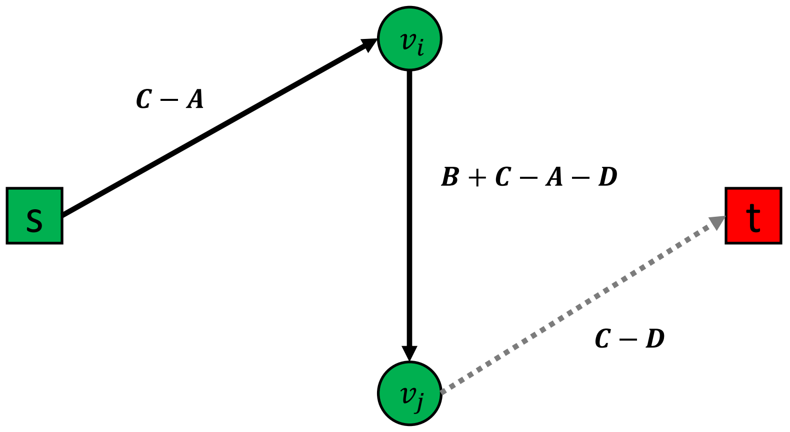
2. The class $\mathcal{F}^2$

- Scenario 1: Minimum cut at edge $(\boldsymbol{v_j}, \boldsymbol{t})$

$E^{i,j}(0{,}0)$	$E^{i,j}(0{,}1)$
$E^{i,j}(1{,}0)$	$E^{i,j}(1{,}1)$

=

A	B
C	D



$$(\boldsymbol{v_j}, \boldsymbol{t}) < (\boldsymbol{s}, \boldsymbol{v_i})$$
$$C - D < C - A$$
$$\boldsymbol{A} < \boldsymbol{D}$$

$$(\boldsymbol{v_j}, \boldsymbol{t}) < (\boldsymbol{v_i}, \boldsymbol{v_j})$$
$$C - D < B + C - A - D$$
$$\boldsymbol{A} < \boldsymbol{B}$$

$$(\boldsymbol{s}, \boldsymbol{v_i}) > \boldsymbol{0}$$
$$C - A > 0$$
$$\boldsymbol{A} < \boldsymbol{C}$$

$$\Rightarrow \boldsymbol{A\ is\ the\ minimum}$$

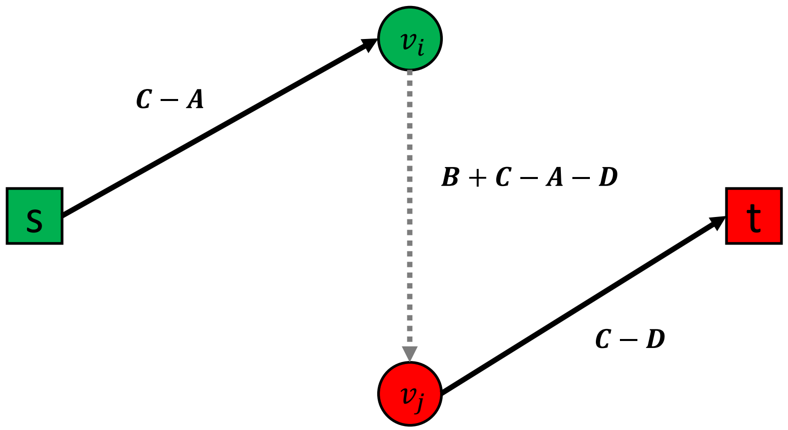
2. The class $\mathcal{F}^2$

- Scenario 2: Minimum cut at edge $(\boldsymbol{v_i}, \boldsymbol{v_j})$

$E^{i,j}(0,0)$	$E^{i,j}(0,1)$
$E^{i,j}(1,0)$	$E^{i,j}(1,1)$

=

A	B
C	D



$$(\boldsymbol{v_i}, \boldsymbol{v_j}) < (\boldsymbol{s}, \boldsymbol{v_i})$$
$$B + C - A - D < C - A$$
$$\boldsymbol{B} < \boldsymbol{D}$$

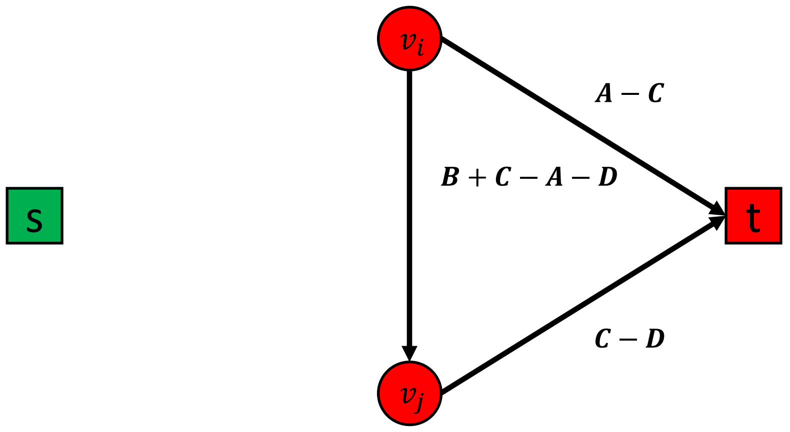
$$(\boldsymbol{v_i}, \boldsymbol{v_j}) < (\boldsymbol{v_j}, \boldsymbol{t})$$
$$B + C - A - D < C - D$$
$$\boldsymbol{B} < \boldsymbol{A}$$

$$(\boldsymbol{s}, \boldsymbol{v_i}) > \boldsymbol{0}$$
$$C - A > 0$$
$$A < C$$
$$\boldsymbol{B} < \boldsymbol{C}$$

$$\Rightarrow \boldsymbol{B\ is\ the\ minimum}$$

2. The class $\mathcal{F}^2$

- Scenario 3



$E^{i,j}(0,0)$	$E^{i,j}(0,1)$
$E^{i,j}(1,0)$	$E^{i,j}(1,1)$

=

A	B
C	D

$$\boldsymbol{(v_j, t) > 0}$$
$$C - D > 0$$
$$\boldsymbol{D < C}$$

$$\boldsymbol{(v_i, t) > 0}$$
$$A - C > 0$$
$$C < A$$
$$\boldsymbol{D < A}$$

$$\boldsymbol{(v_i, v_j) > 0}$$
$$B + C - A - D > 0$$
$$D + (A - C) < B$$
$$\boldsymbol{D < B}$$

$$\Rightarrow \boldsymbol{D\ is\ the\ minimum}$$

3. The class $\mathcal{F}^3$

- Functions that can be written as a **sum of functions of up to three variables**

$$E(x_1, \dots, x_n) = \sum_i E^i(x_i) + \sum_{i<j} E^{i,j}(x_i, x_j) + \sum_{i<j<k} E^{i,j,k}(x_i, x_j, x_k)$$

- E is graph-representable **all functions of two variables** are **regular**

$$E^{i,j}(0,0) + E^{i,j}(1,1) \leq E^{i,j}(0,1) + E^{i,j}(1,0)$$

- E is graph-representable if **all functions of more than two variables** are regular.

  Such functions are regular if all of their **projections** are **regular**.

3. The class $\mathcal{F}^3$

- The concept of **projections**:

For a function of n binary variables	$\boldsymbol{E(x_1, \dots, x_n)}$,
a disjoint partition of the indices	$\boldsymbol{(1, \dots, n): I = \{i(1), \dots, i(m)\},\ J = \{j(1), \dots, j(n-m)\}}$,
and a set of binary constants	$\boldsymbol{\alpha_{i(1)}, \dots, \alpha_{i(m)}}$,
the function E' of $n-m$ variables	$\boldsymbol{E'(x_{j(1)}, \dots, x_{j(n-m)}) = E[x_{i(1)} = \alpha_{i(1)}, \dots, x_{i(m)} = \alpha_{i(m)}]}$
is a projection of E	$\boldsymbol{E'(x_{j(1)}, \dots, x_{j(n-m)}) = E(x_1, \dots, x_n)}$

- We say that we have fixed the variables $\boldsymbol{x_{i(1)} = \alpha_{i(1)}, \dots, x_{i(m)}}$ where $\boldsymbol{x_i = \alpha_i}$ for $\boldsymbol{i \in I}$

3. The class $\mathcal{F}^3$

- A term $E^{i,j,k}$ depending on **three variables** x_i, x_j and x_k can be rewritten as follows:

$\boldsymbol{E^{i,j,k}}$ =

$\boldsymbol{E^{i,j,k}(0,0,0)}$	$\boldsymbol{E^{i,j,k}(0,0,1)}$
$\boldsymbol{E^{i,j,k}(0,1,0)}$	$\boldsymbol{E^{i,j,k}(0,1,1)}$
$\boldsymbol{E^{i,j,k}(1,0,0)}$	$\boldsymbol{E^{i,j,k}(1,0,1)}$
$\boldsymbol{E^{i,j,k}(1,1,0)}$	$\boldsymbol{E^{i,j,k}(1,1,1)}$

=

$\boldsymbol{A}$	$\boldsymbol{B}$
$\boldsymbol{C}$	$\boldsymbol{D}$
$\boldsymbol{E}$	$\boldsymbol{F}$
$\boldsymbol{G}$	$\boldsymbol{H}$

3. The class $\mathcal{F}^3$

- Expansion of $E^{i,j,k}$

  (for the case $P > 0$)

$$
\begin{bmatrix} A & B \\ C & D \\ E & F \\ G & H \end{bmatrix}
= A +
\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ P_1 & P_1 \\ P_1 & P_1 \end{bmatrix}
+
\begin{bmatrix} 0 & 0 \\ P_2 & P_2 \\ 0 & 0 \\ P_2 & P_2 \end{bmatrix}
+
\begin{bmatrix} 0 & P_3 \\ 0 & P_3 \\ 0 & P_3 \\ 0 & P_3 \end{bmatrix}
+
\begin{bmatrix} 0 & P_{23} \\ 0 & 0 \\ 0 & P_{23} \\ 0 & 0 \end{bmatrix}
+
\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ P_{31} & 0 \\ P_{31} & 0 \end{bmatrix}
+
\begin{bmatrix} 0 & 0 \\ P_{12} & P_{12} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
+
\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -P \end{bmatrix}
$$

3. The class $\mathcal{F}^3$

- Unary terms of $E^{i,j,k}$

$\mathbf{0}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$
$\boldsymbol{P_1}$	$\boldsymbol{P_1}$
$\boldsymbol{P_1}$	$\boldsymbol{P_1}$

+

$\mathbf{0}$	$\mathbf{0}$
$\boldsymbol{P_2}$	$\boldsymbol{P_2}$
$\mathbf{0}$	$\mathbf{0}$
$\boldsymbol{P_2}$	$\boldsymbol{P_2}$

+

$\mathbf{0}$	$\boldsymbol{P_3}$
$\mathbf{0}$	$\boldsymbol{P_3}$
$\mathbf{0}$	$\boldsymbol{P_3}$
$\mathbf{0}$	$\boldsymbol{P_3}$

$$P_1 = F - B = E^{i,j,k}(\mathbf{1},0,1) - E^{i,j,k}(\mathbf{0},0,1)$$

$$P_2 = G - E = E^{i,j,k}(1,\mathbf{1},0) - E^{i,j,k}(1,\mathbf{0},0)$$

$$P_3 = D - C = E^{i,j,k}(0,1,\mathbf{1}) - E^{i,j,k}(0,1,\mathbf{0})$$

3. The class $\mathcal{F}^3$

- Binary terms of $E^{i,j,k}$

0	$\boldsymbol{P_{23}}$
0	**0**
0	$\boldsymbol{P_{23}}$
0	**0**

+

0	**0**
0	**0**
$\boldsymbol{P_{31}}$	**0**
$\boldsymbol{P_{31}}$	**0**

+

0	**0**
$\boldsymbol{P_{12}}$	$\boldsymbol{P_{12}}$
0	**0**
0	**0**

$$
\begin{aligned}
P_{23} &= B + C - A - D \\
&= E^{i,j,k}(0,\mathbf{0},\mathbf{1}) + E^{i,j,k}(0,\mathbf{1},\mathbf{0}) - E^{i,j,k}(0,\mathbf{0},\mathbf{0}) - E^{i,j,k}(0,\mathbf{1},\mathbf{1})
\end{aligned}
$$

$$
\begin{aligned}
P_{31} &= B + E - A - F \\
&= E^{i,j,k}(\mathbf{0},0,\mathbf{1}) + E^{i,j,k}(\mathbf{1},0,\mathbf{0}) - E^{i,j,k}(\mathbf{0},0,\mathbf{0}) - E^{i,j,k}(\mathbf{1},0,\mathbf{1})
\end{aligned}
$$

$$
\begin{aligned}
P_{12} &= C + E - A - G \\
&= E^{i,j,k}(\mathbf{0},\mathbf{1},0) + E^{i,j,k}(\mathbf{1},\mathbf{0},0) - E^{i,j,k}(\mathbf{0},\mathbf{0},0) - E^{i,j,k}(\mathbf{1},\mathbf{1},0)
\end{aligned}
$$

3. The class $\mathcal{F}^3$

- Ternary term of $E^{i,j,k}$
- An **auxiliary vertex $\boldsymbol{u_{ijk}}$** is added

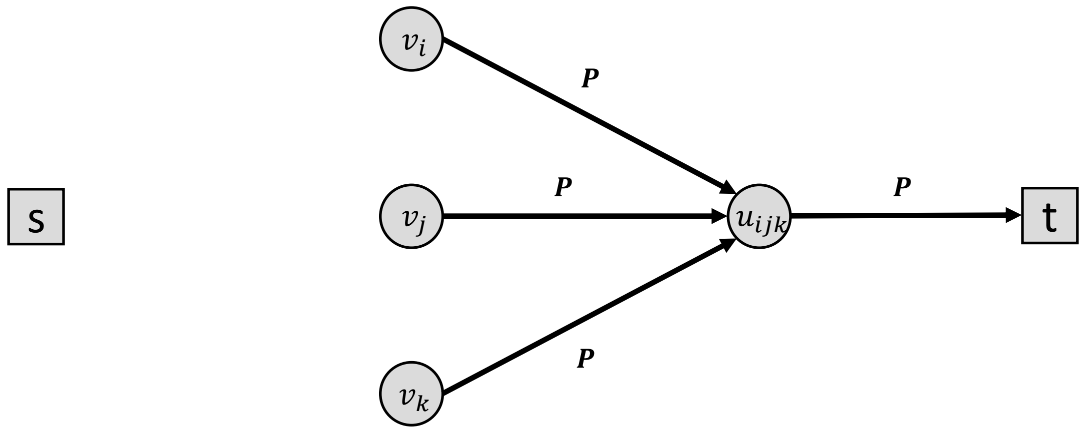
0	**0**
0	**0**
0	**0**
0	$-\boldsymbol{P}$

$+ \ \boldsymbol{P} \ =$

$\boldsymbol{P}$	$\boldsymbol{P}$
$\boldsymbol{P}$	$\boldsymbol{P}$
$\boldsymbol{P}$	$\boldsymbol{P}$
$\boldsymbol{P}$	**0**

Constant function

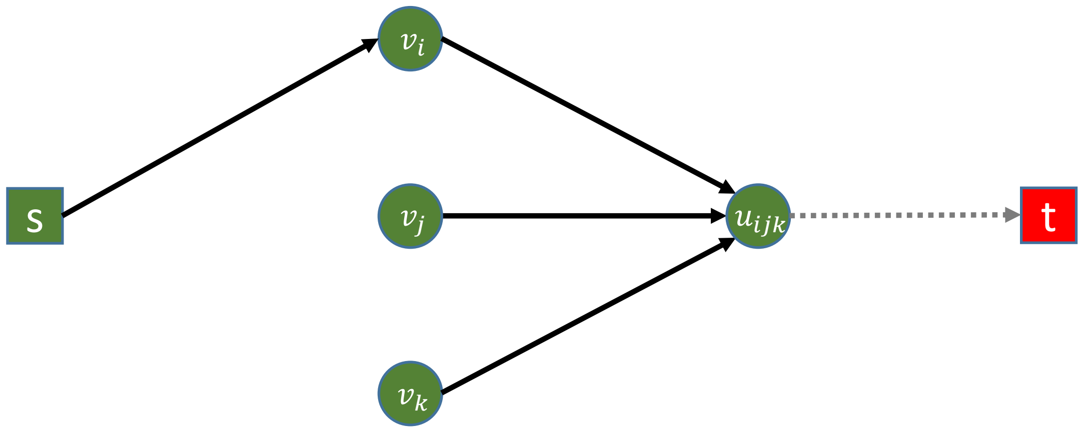
We want to represent this!



$$P = (A + D + F + G) - (B + C + E + H)$$

3. The class $\mathcal{F}^3$

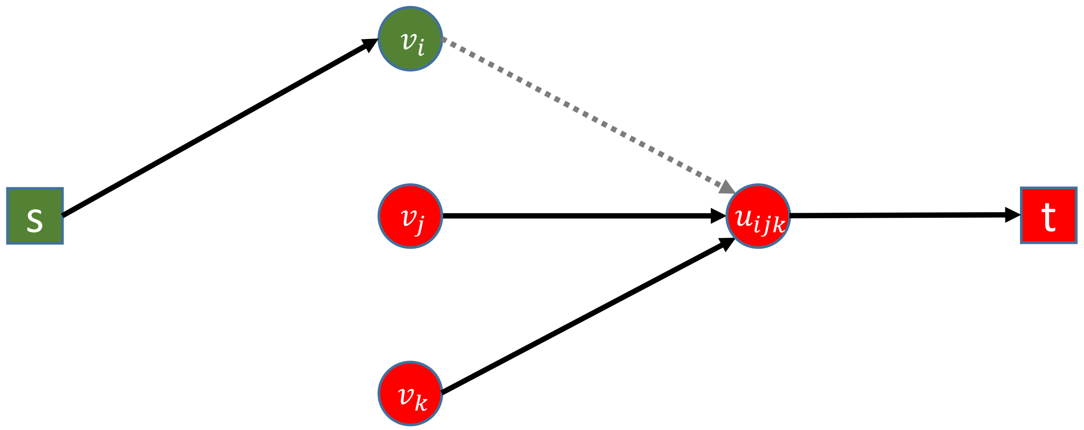
- Scenario 1: $\boldsymbol{v_i} \in \boldsymbol{S}$ and $\boldsymbol{u_{i,j,k}} \in \boldsymbol{S}$



$\Rightarrow \boldsymbol{the\ minimum\ cut\ is\ P}$

3. The class $\mathcal{F}^3$

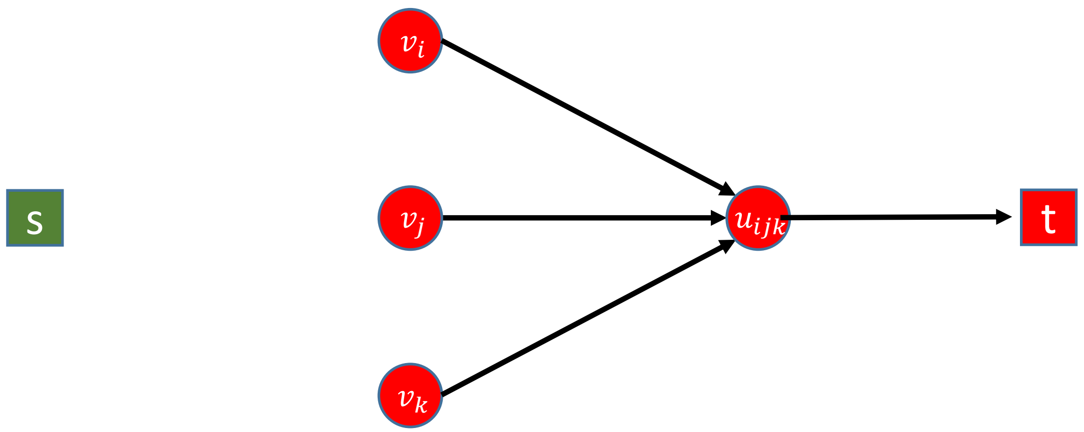
- Scenario 2: $\boldsymbol{v_i} \in \boldsymbol{S}$ and $\boldsymbol{u_{i,j,k}} \in \boldsymbol{T}$



$\Rightarrow \boldsymbol{the\ minimum\ cut\ is\ P}$

3. The class $\mathcal{F}^3$

- Scenario 3: $\boldsymbol{x_i = x_j = x_k = 1}$



$\Rightarrow \boldsymbol{the\ minimum\ cut\ is\ 0}$

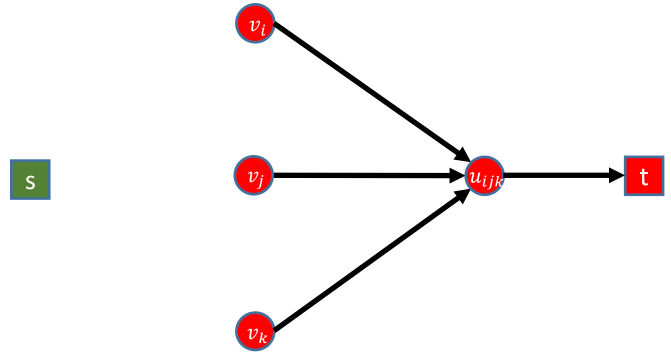
3. The class $\mathcal{F}^3$

- Hence, it is shown that the cost of the minimum cut will always be **P**, except for the case $\boldsymbol{x_i = x_j = x_k = 1}$

$\mathbf{0}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$
$\mathbf{0}$	$\boldsymbol{-P}$

$+ \ \boldsymbol{P} \ =$

$\boldsymbol{P}$	$\boldsymbol{P}$
$\boldsymbol{P}$	$\boldsymbol{P}$
$\boldsymbol{P}$	$\boldsymbol{P}$
$\boldsymbol{P}$	$\mathbf{0}$



$\boldsymbol{E^{i,j,k}(0,0,0)}$	$\boldsymbol{E^{i,j,k}(0,0,1)}$
$\boldsymbol{E^{i,j,k}(0,1,0)}$	$\boldsymbol{E^{i,j,k}(0,1,1)}$
$\boldsymbol{E^{i,j,k}(1,0,0)}$	$\boldsymbol{E^{i,j,k}(1,0,1)}$
$\boldsymbol{E^{i,j,k}(1,1,0)}$	$\boldsymbol{E^{i,j,k}(1,1,1)}$

4. Regularity

- **If a function of binary variables is not regular, it is not graph-representable**
- First, a more convenient definition of graph representability:

Graph representability

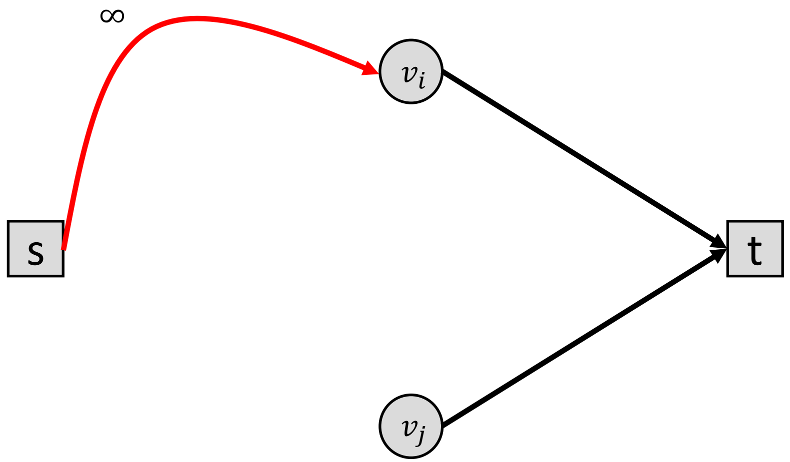
$G = (V, \mathcal{E})$ is a graph,
$v_1, \ldots, v_n$ is a subset of V, and
$\alpha_1, \ldots, \alpha_k$ is a set of binary constants with values $\{0, 1\}$.

$G[x_1 = \alpha_1, \ldots, x_k = \alpha_k]$ will be the same as in G,
plus additonal edges with **infinite capacities** corresponding to $v_1, \ldots, v_k$,
where (s, v_i) is added if $\alpha_i = 0$ or (v_i, t) if $\alpha_i = 1$.

E is exactly represented by G if for any configuration $\alpha_1, \ldots, \alpha_n$,
the minimum cut on $G[x_1 = \alpha_1, \ldots, x_k = \alpha_k]$ = $E[\alpha_1, \ldots, \alpha_n]$.

4. Regularity

- The edges with infinite capacities impose **constraints on the minimum cut** of $G[x_1 = \alpha_1, \dots, x_k = \alpha_k]$
- For example: $\boldsymbol{\alpha_1 = 0}$ and $\boldsymbol{v_i} \in \boldsymbol{T}$



- $(\boldsymbol{s}, \boldsymbol{v_i})$ is prohibited from being the minimum cut, as cutting it yields an infinite cost

4. Regularity

- Let us prove that regularity is a necessary condition for graph-representability
- Consider a graph-representable function $\bar{E}(x_1, x_2)$

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & A \\ \hline \end{array} = \bar{E}(0,0) + \begin{array}{|c|c|}\hline 0 & -\bar{E}(0,1) \\ \hline 0 & -\bar{E}(0,1) \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline -\bar{E}(1,0) & -\bar{E}(1,0) \\ \hline \end{array} + \begin{array}{|c|c|}\hline \bar{E}(0,0) & \bar{E}(0,1) \\ \hline \bar{E}(1,0) & \bar{E}(1,1) \\ \hline \end{array}$$

- Each function on the right hand side is graph-representable
- Hence, the function on the left hand side is graph-representable as well *(Additivity theorem)*

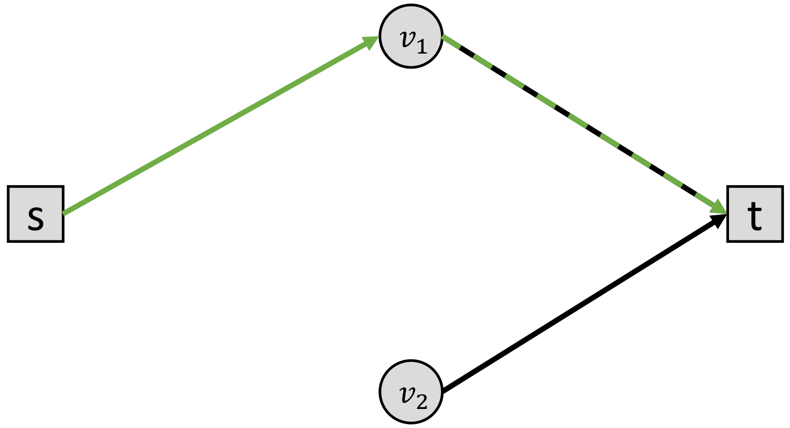
Let $E = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & A \\ \hline \end{array}$ be represented by the graph $\boldsymbol{G}$

$$A = \bar{E}(0,0) + \bar{E}(1,1) - \bar{E}(0,1) - \bar{E}(1,0)$$

$$\boldsymbol{A \leq 0}$$

4. Regularity

- **Let $A > 0$.**
- The minimum cut and maximum flow of G is 0. There is no augmenting path from s to t
- Let us add the edges (v_1, t) and (v_2, t) to G. There must be an augmenting path from s to t to satisfy $E(1,1) > 0$



E =

0	0
0	A

=

$E(0,0)$	$E(0,1)$
$E(1,0)$	$E(1,1)$

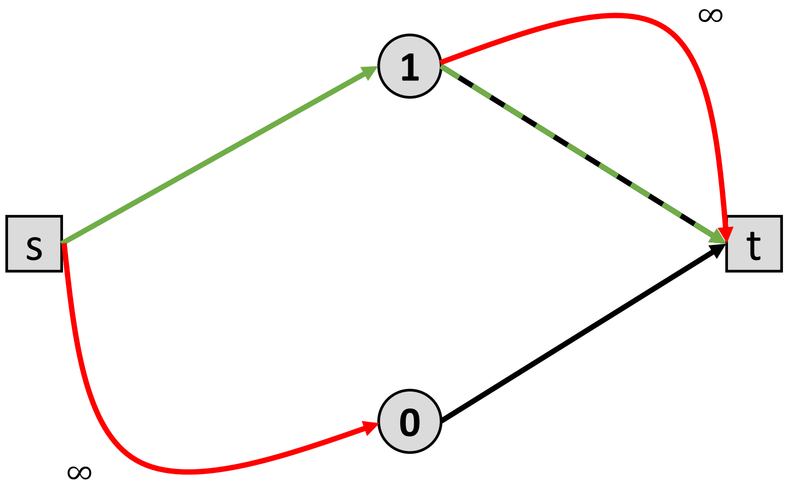
Added edge

Augmenting path

Infinite edge

4. Regularity

- Let $G[x_1 = 1, x_2 = 0]$ by adding edges (v_1, t) and (s, v_2) with infinite capacities
- There exists an augmenting path $\{P, (v_1, t)\}$ from s to t
- The maximum flow (minimum cut) is therefore greater than 0, meaning $E(1, 0) > 0$
- We get a **contradiction!**

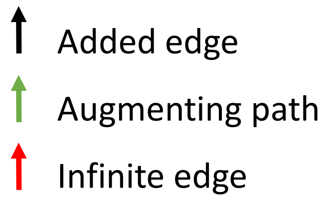


E =

0	0
0	A

=

$E(0,0)$	$E(0,1)$
$E(1,0)$	$E(1,1)$



5. Summary

- Shown how energy functions can be represented as graphs, where the minimum s-t cut minimizes the energy
- Presented the class $\mathcal{F}^2$ for functions of up to two binary variables and a means of graph construction
- Presented the class $\mathcal{F}^3$ for functions of up to three binary variables and a means of graph construction
- Shown that regularity is a necessary condition for graph-representability

- Based on the paper by V. Kolmogorov and R. Zabin, "What energy functions can be minimized via graph cuts?," in *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 2004.

Questions?

Thanks for your attention! ☺