Generalized Multidimensional Scaling

Isometry-invariant partial surface matching

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November 23, 2016

Goal

- The goal of this algorithm is to compute the the minimum-distortion mapping between two surfaces.
- A practical use would be for face recognition

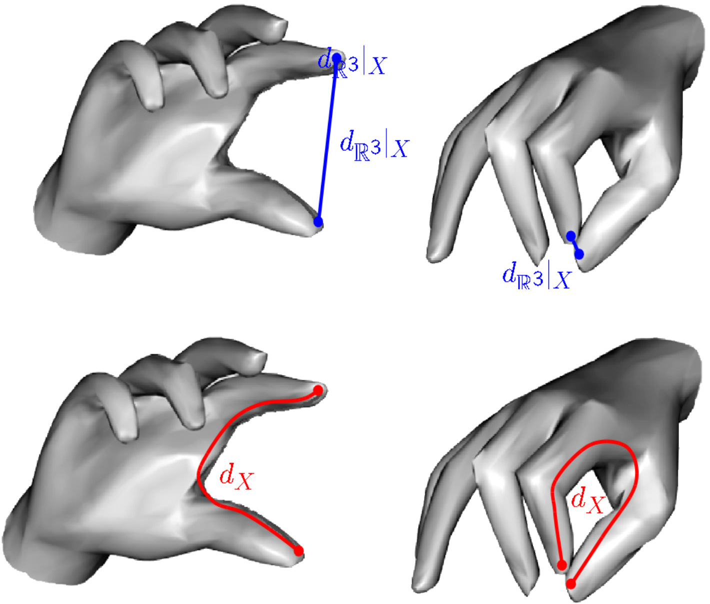
Metric Space

Definition

A **Metric Space** is an ordered pair (M, d), where M is a set and d is a function $d : M \times M \to \mathbb{R}$ with the following properties:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \Leftrightarrow x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

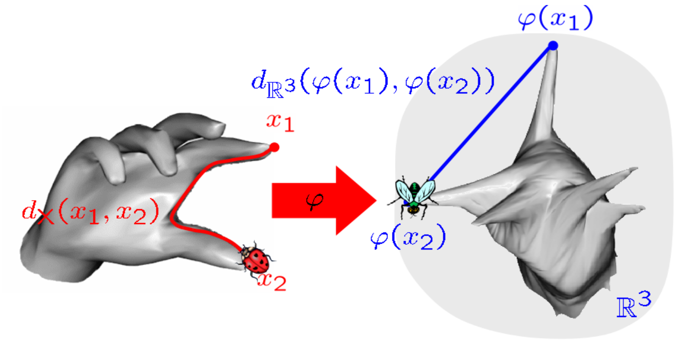
Metric Space



Isometry

Definition

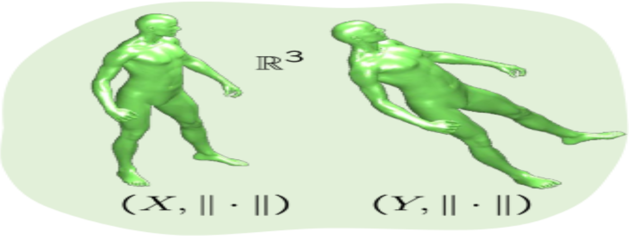
Let X and Y be two metric spaces with metrics d_X and d_Y. A function $\varphi : X \to Y$ is called an isometry if for every $a, b \in X$ $d_X(a, b) = d_Y(\varphi(a), \varphi(b))$



Iterative Closest Point

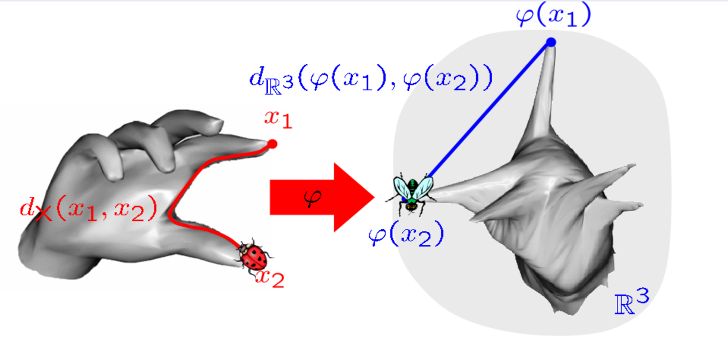
Definition

Iterative Closest Point is an algorithm used to determine the best rigid motion **(R, t)** in order to align two shapes. Basically, it is a matching method where only **Euclidean** isometries are allowed.



Multidimensional scaling (MDS)

Canonical Form



Multidimensional scaling (MDS)

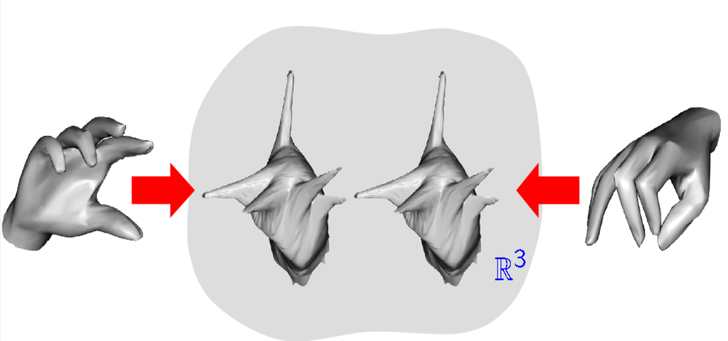
Definition

- **Multidimensional Scaling** is an algorithm that determines the level of similarity between two shapes by minimizing the following *stress* function for both shapes.

- $\sigma(\phi; D_{S_N}) = \frac{1}{N} \cdot \sum_{i>j} (d_{\mathbb{R}^m}(\phi(s_i), \phi(s_j)) - d_S(s_i, s_j))^2$

Multidimensional scaling (MDS)

Canonical Form



A distance between shapes

Metric properties

1. $d(S,Q) \geq 0$
2. $d(S,Q) = 0 \Leftrightarrow S$ and Q are isometric
3. $d(S,Q) = d(Q,S)$
4. $d(R,Q) \leq d(R,S) + d(S,Q)$

Good similarity measure

- $d(S,Q) \leq \epsilon \Rightarrow S$ and Q are $\epsilon -$ *isometric*

A distance between shapes

Partial matching

- Given S' a patch of S then $d(S', S) = 0$

Sampling consistency

- Given S^r a finite r-covering of S, $lim_{r \to 0} d(S^r, Q) = d(S, Q)$

Efficiency

- d should be efficiently computable, or at least efficiently approximated

Gromov Hausdorff distance

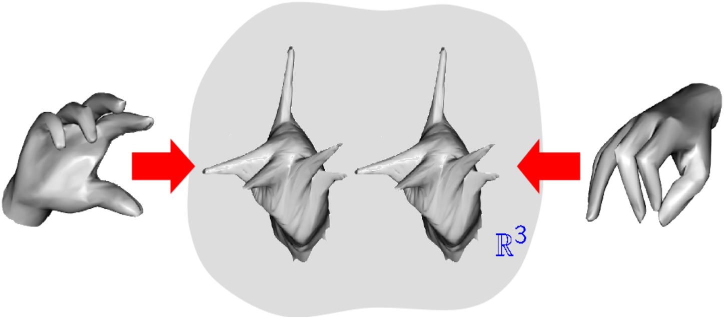
Hausdorff distance

- The **Hausdorff** distance measures how far two subsets of a metric space are from each other.
- $d_H(X, Y) = max(\sup_{x \in X} \inf_{y \in Y} d(y, x), \sup_{y \in Y} \inf_{x \in X} d(x, y))$

Gromov Hausdorff distance

Gromov Hausdorff distance

- The **Gromov Hausdorff** distance measures how far two compact metric spaces are from being isometric.
- $d_{GH}(S,Q) = \inf_{Z,\phi,\psi} d_H^Z(\phi(S),\psi(Q))$



General Multidimensional Scaling

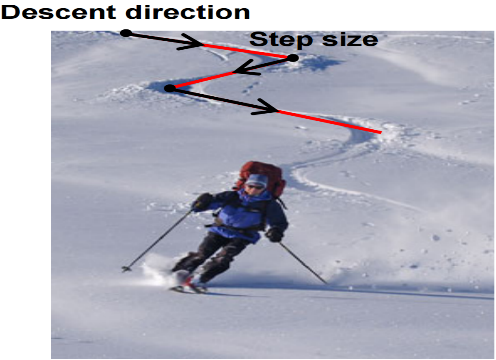
Concept

- The **Gromov Hausdorff** distance takes into account all possible metric spaces for mapping the canonical form.
- **GMDS** takes only the second space in order to project the Canonical Form
- $d_{GH}(S, Q) = \inf d_H^Q(\phi(S), Q)$

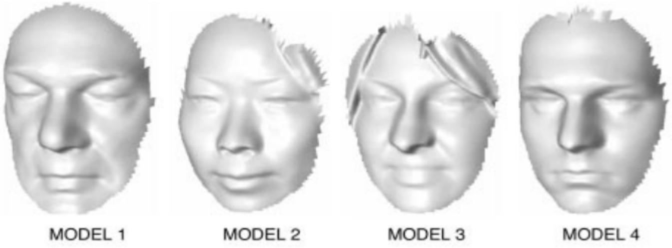
General Multidimensional Scaling

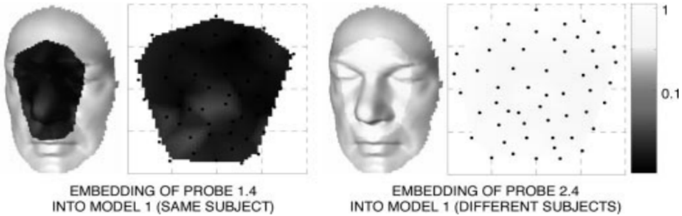
Algorithm

- $\sigma_p(\mathbf{U}; D_{Q_N}; d_S) = (\sum_{j>i}((d_S(u_i, u_j) - d_Q(q_i, q_j)))^p)^{\frac{1}{p}}$
- $U^{(k+1)} = P_U(U^k - \alpha^k \nabla_U \sigma_p(U^{(k)}))$



Results


MODEL 1
MODEL 2
MODEL 3
MODEL 4


1
0.1
EMBEDDING OF PROBE 1.4
INTO MODEL 1 (SAME SUBJECT)
EMBEDDING OF PROBE 2.4
INTO MODEL 1 (DIFFERENT SUBJECTS)

Questions?