A Concise and Provably Informative Multi-Scale Signature Based on Heat Diffusion

Jian Sun, Maks Ovsjanikov, Leonidas Guibas (2009)

Presented by: Julia Fokuhl

Technische Universität München

Image Segmentation and Shape Analysis

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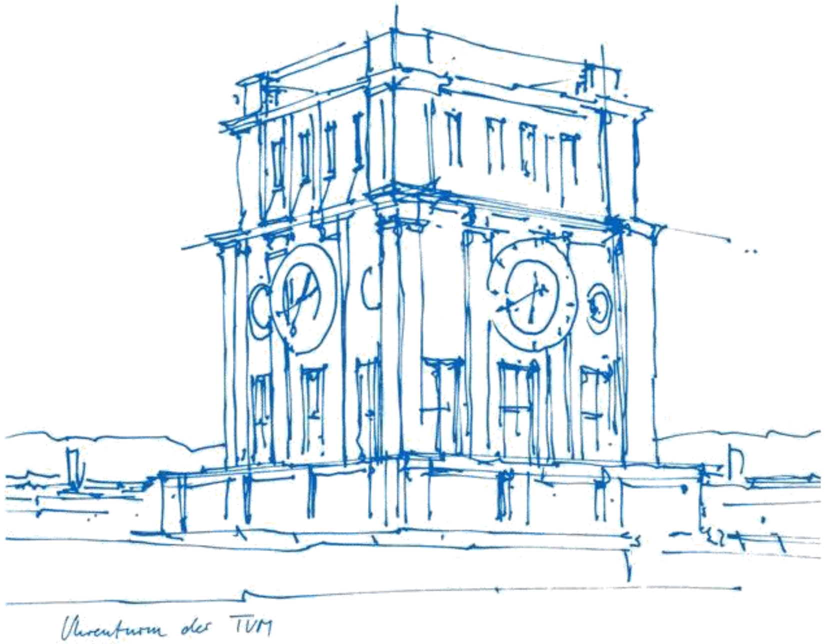
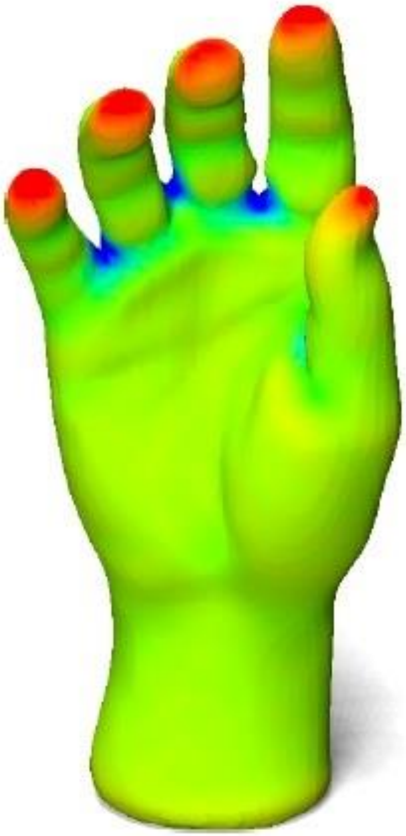


Motivation

Characterise shapes

Gain information about the geometry

Identify features

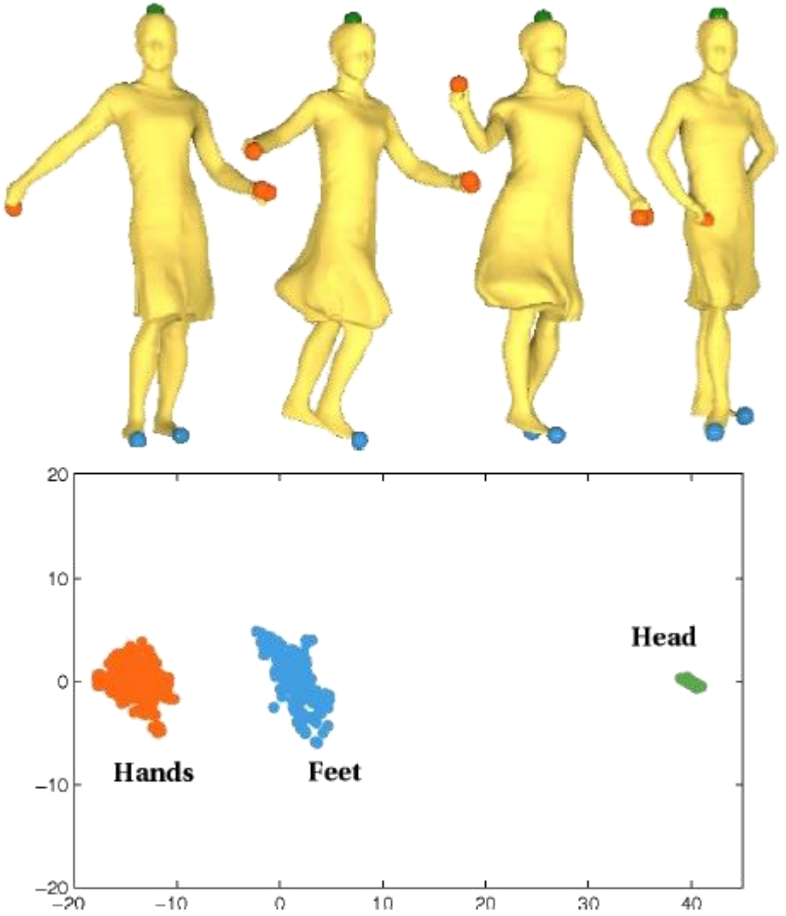


Motivation - Applications

Shape Comparison/ Matching shapes

Detect repeated structures
(across one or more shapes)

Classification



Heat Kernel Signature (HKS)

Point signature based on heat diffusion
Preserves information about intrinsic geometry

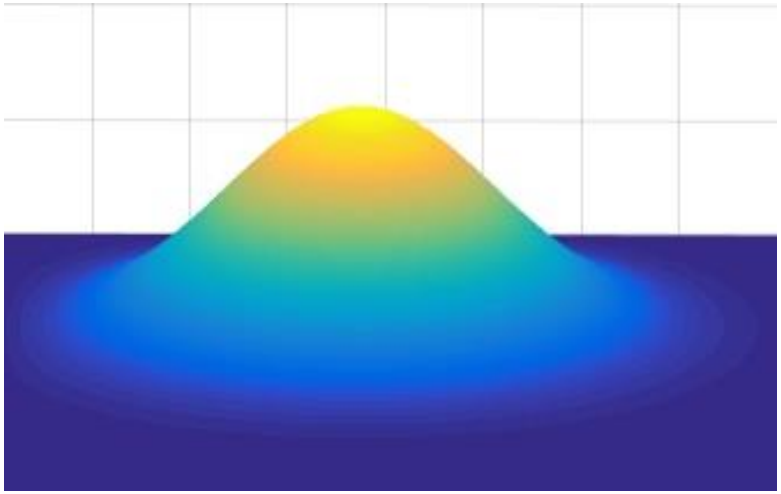
Properties:

- Efficient calculation
- Concise
- Multi-scale
- Stable
- Invariant under isometric deformations

Heat Diffusion

on a compact Riemannian manifold M:

$$\Delta u(x,t) = -\frac{\partial u(x,t)}{\partial t}$$



with

Δ : Laplace-Beltrami-Operator

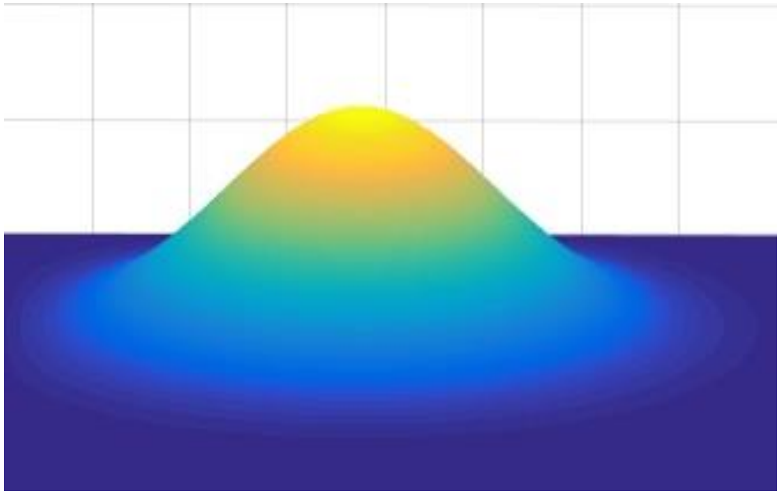
$u(x,t)$: heat distribution at point x and time t

If M has boundaries, the Dirichlet boundary condition has to be fulfilled:

$u(x,t) = 0 \; for \; all \; x \in \partial M \; and \; all \; t$

Heat Diffusion

$$\Delta u(x,t) = -\frac{\partial u(x,t)}{\partial t}$$



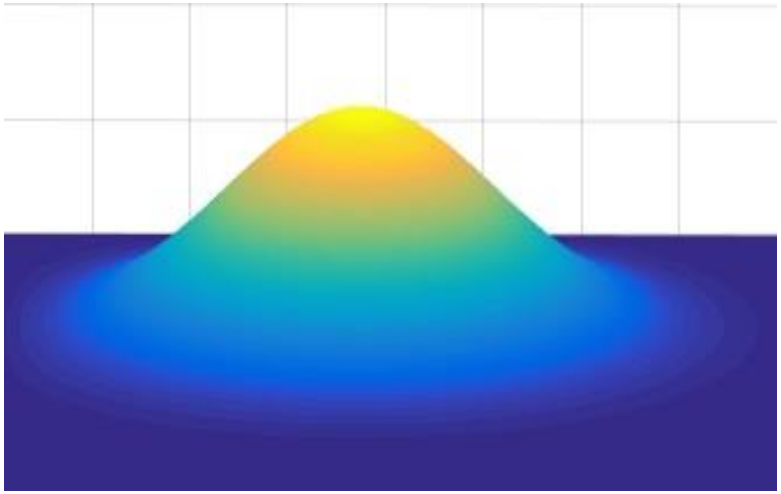
Initial heat distribution $f : M \to \mathbb{R}$
Heat distribution after time t
→ Heat Operator: $H_t(f)$

Relation for the operators: $H_t = e^{-t\Delta}$
thus both operators H_t and Δ share the same eigenfunctions

Heat Kernel

$$\Delta u(x,t) = -\frac{\partial u(x,t)}{\partial t}$$

$$H_t f(x) = \int_M k_t(x,y) f(y) dy$$



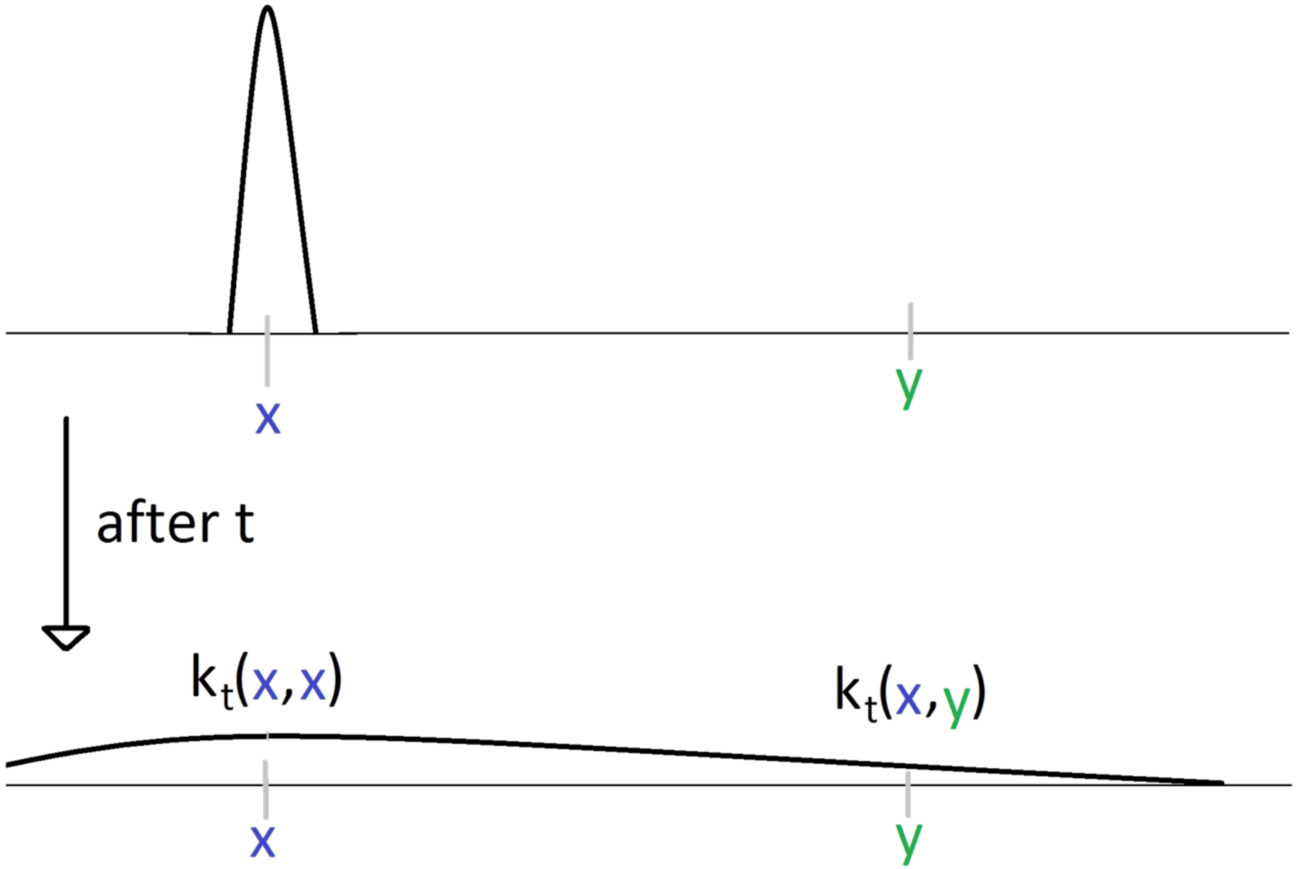
dy: volume form at $y \epsilon M$

$k_t(x,y)$: heat kernel (amount of heat transferred from x to y in time t given a unit heat source at x)

$k_t(x,\cdot) = H_t(\delta_x)$ with δ_x: Dirac-Delta

Heat Kernel

amount of heat transferred from x to y in time t given a unit heat source at x



Heat Kernel

can be represented in the Laplace-Beltrami eigenbasis

Eigendecomposion:

$$k_t(x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x) \phi_i(y)$$

$\phi_i(x)$: i^{th} eigenfunction of the Laplace-Beltrami-Operator

λ_i: i^{th} eigenvalue of the Laplace-Beltrami-Operator

Repetition: Intrinsic Isometry

Mapping $\Phi: M \rightarrow N$ between two shapes M, N is an isometry, if the geodesic distances are preserved

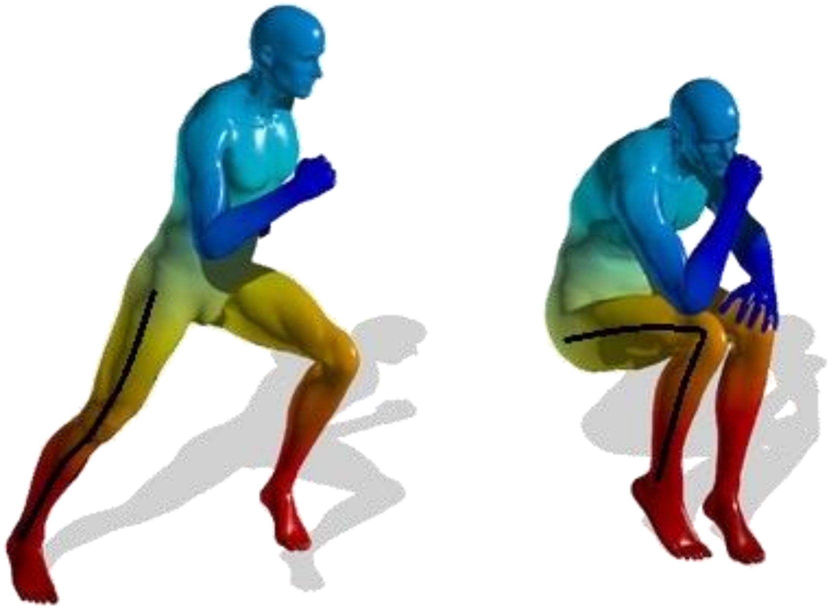


Image adapted from:
Shtern, A.; Kimmel, R. Matching the LBO Eigenspace of Non-Rigid Shapes via High Order Statistics. *Axioms* **2014**, *3*, 300-319.

Heat Kernel

is intrinsic → is an isometric invariant

$If\ \Phi : M \ \rightarrow N\ is\ an\ isometry\ between\ Riemannian\ manifolds\ M\ and\ N,$
$then\ k_{\mathrm{t}}^{M}(x,y) = k_t^N\big(\Phi(x),\Phi(y)\big)\ for\ any\ x,y \in M\ and\ t > 0$

is informative → fully characterises shapes up to isometry

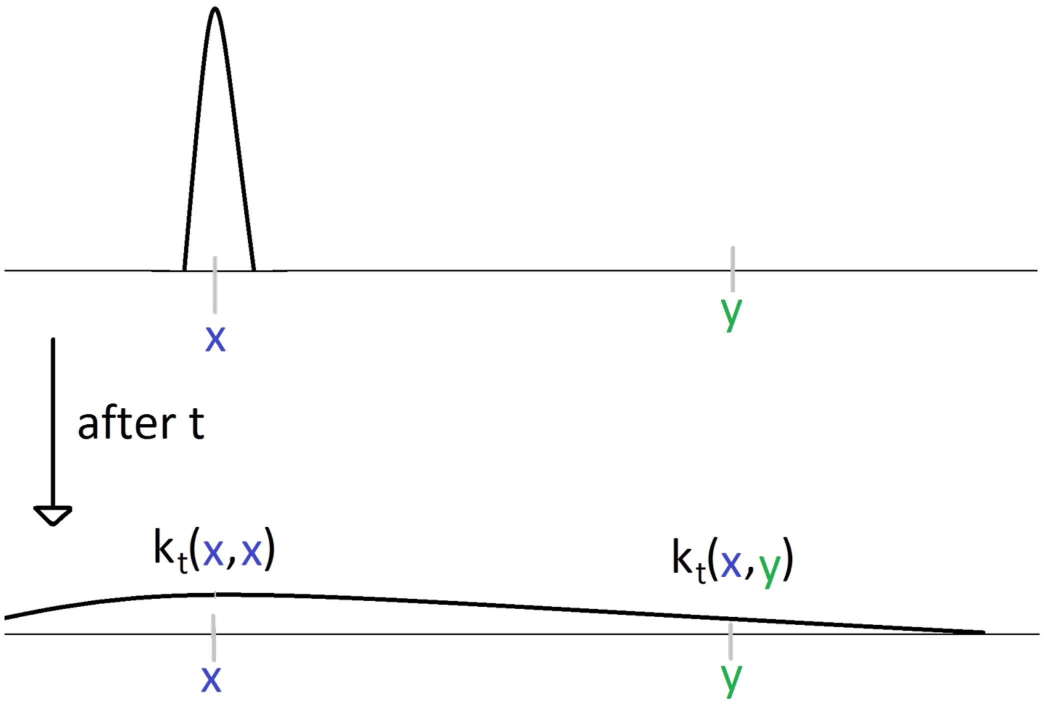
$Let\ \Phi\text{:}\ M \ \rightarrow N\ be\ surjective\ map\ between\ Riemannian\ manifolds\ M\ and\ N$
$If\ k_{\mathrm{t}}^{M}(x,y) = k_t^N\big(\Phi(x),\Phi(y)\big)\ for\ any\ x,y \in M\ and\ t > 0, then\ \Phi\ is\ an\ isometry$

Heat Kernel

Family of functions $\{ k_t(x,\cdot) \}_{t>0}$

High complexity:
defined on $\mathbb{R}^+ \times M$

Redundant information
→ Heat Kernel Signature



$$HKS(x,t) = k_t(x,x)$$

for a point x on the manifold M: $HKS(x)$: $\mathbb{R}^+ \rightarrow \mathbb{R}$,

Heat Kernel Signature

$HKS(x, t) = k_t(x, x)$

Dropping spatial domain

$\{ k_t(x, x) \}_{t>0}$ is almost as informative as $\{ k_t(x, \cdot) \}_{t>0}$

HKS at different points are defined on a common temporal domain

→ Easily commensurable

Properties:

- ✓ Efficient calculation
- ✓ Concise
- • Multi-scale
- • Stable
- • Invariant under isometric deformations

Time as scaling parameter

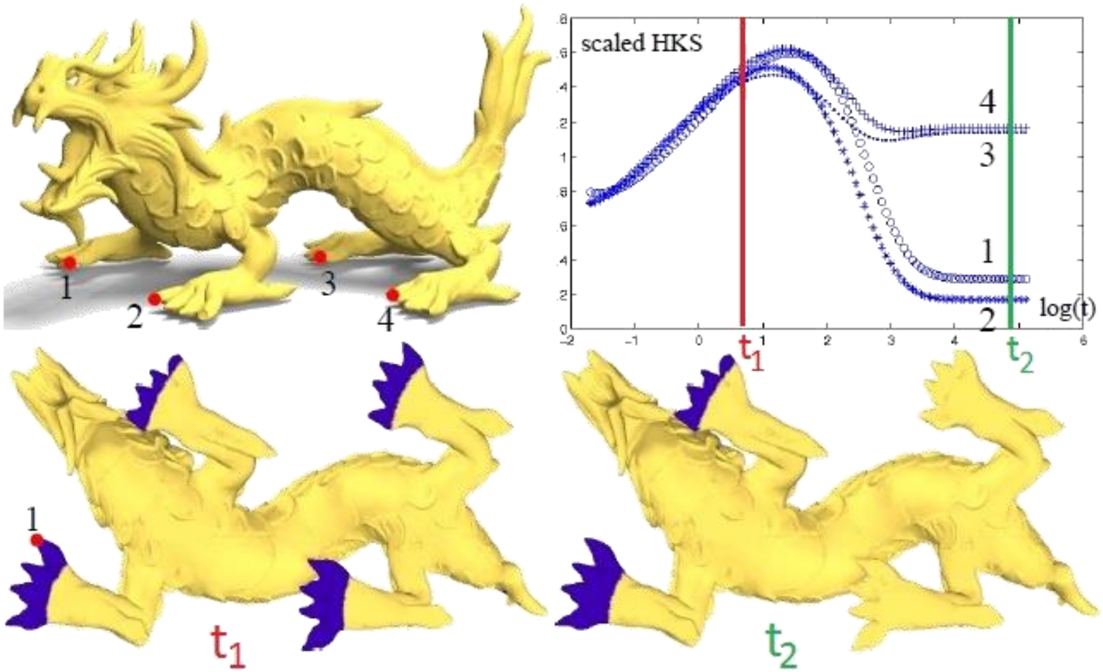
Dissipation of heat from a point x

Heat diffuses progressively to larger neighbourhoods

Highly local features/
Small Neighbourhood
– observed at short times

Big features/ Global Structure
– observed after long times

→ Time as scaling parameter
→ Multi-scale Signature



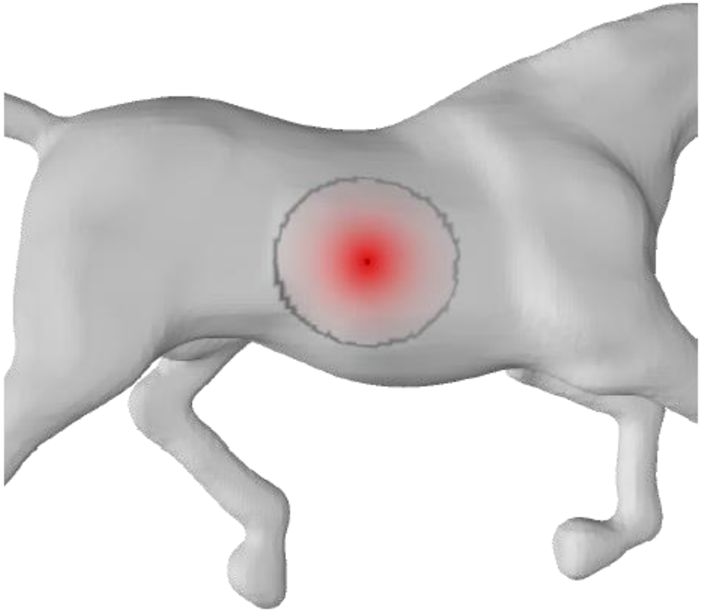
Multiscale

Subset $D \subseteq M$ (D: grey circle on horse)
$k_t^D(x,y)$ is a good approximation of $k_t(x,y)$ in case that

- D is arbitrary small as long as t is small

$$\lim_{t \to 0} k_t^D(x,y) = k_t(x,y)$$

- t is arbitrary large as long as D is big



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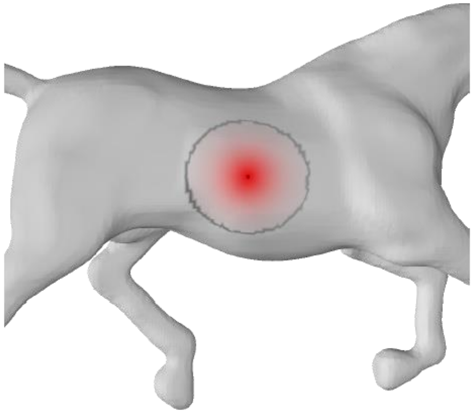

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HKS in discrete settings

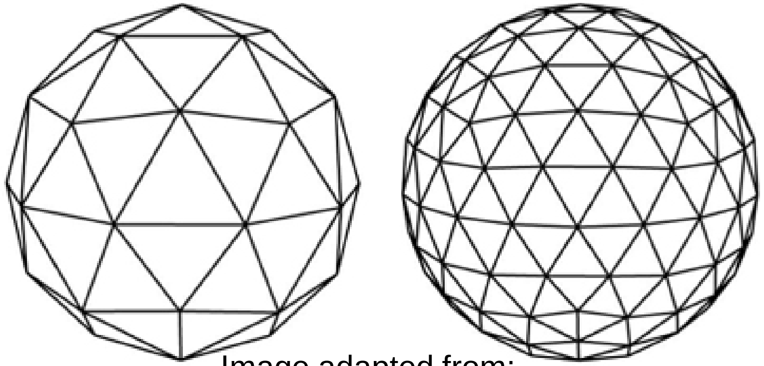


Image adapted from:
http://hessan.annahid.com/images/gdd/geosphere.gif

Underlying manifold unknown

Approximation by mesh with n vertices

Estimation of the Laplace-Beltrami Operator: Mesh Laplace Operator L[1]

$$L = A^{-1}W$$

L: sparse matrix of size $n \times n$

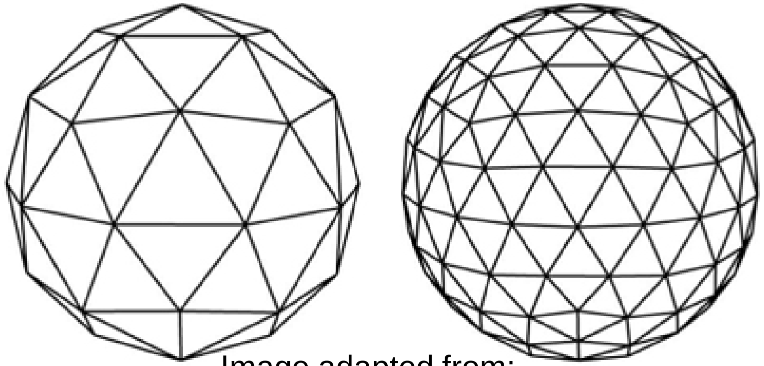
A: positive diagonal matrix

$A(i, i)$ represents an area associated with vertex i

W: symmetric semi-definite matrix

[1]: Belkin M., Sun J., Wang Y.: Discrete Laplace operator on meshed surfaces. In *Proceedings of SOGC* (2008), pp.278-287

HKS in discrete settings



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Heat Equation on the mesh:

$$Lu_t = -\frac{\partial u_t}{\partial t}$$

Reminder for continuous case:

$$\Delta u(x,t) = -\frac{\partial u(x,t)}{\partial t}$$

$u_t(x)$: time-dependent function defined on a vertex x at time t

HKS in discrete settings

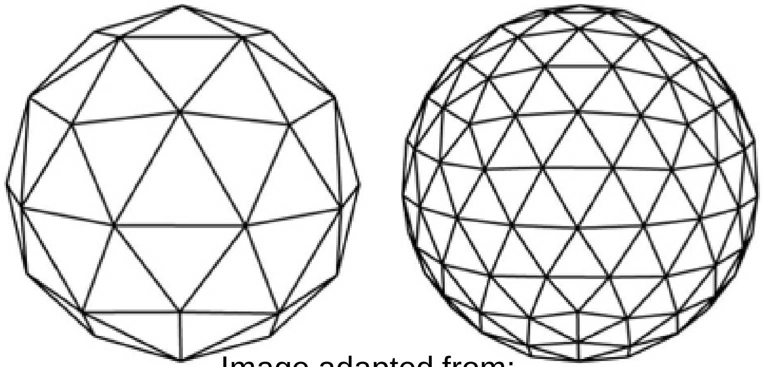


Image adapted from:
http://hessan.annahid.com/images/gdd/geosphere.gif

$$u_t = e^{-tL} f$$

$$u_t(x) = \sum_y k_t(x, y) f(y) A(y)$$

$$k_t(x, y) = \sum_{i=0}^{n} e^{-\lambda_i t} \phi_i(x) \phi_i\,(y)$$

Matching Points

HKS encodes information about the neighbourhood of a point

Matching → Compare HKS of two points in a specific interval $[t_1, t_2]$

$$\left(\int_{t_1}^{t_2} |k_t(x,x) - k_t(x',x')|^2 dt\right)^{1/2}$$

Successful match if expression is approximately 0

Matching Points

Heuristic adaptions:

$|k_t(x,x) - k_t(x',x')|$ decreases with increasing t

→ At small scales differences seem larger

Reminder: $k_t(x,x) = \sum_i e^{-\lambda_i t} \phi_i^2(x)$

Use normalisation:

Scale with $\int_M k_t(x,x)\, dx$

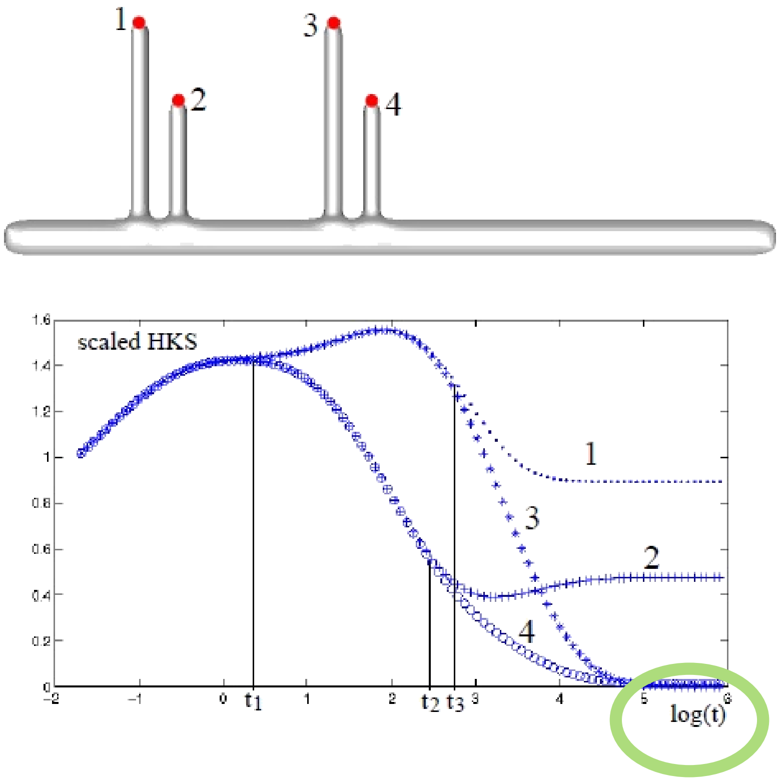
Matching Points

Heuristic adaptions:

$k_t(x,x)$ varies strongly between small times
slower variation for larges times

Intuitive explanation:
HKS changes more rapidly at small scales,
large neighbourhoods are rather stable

→ Scale temporal domain logarithmically

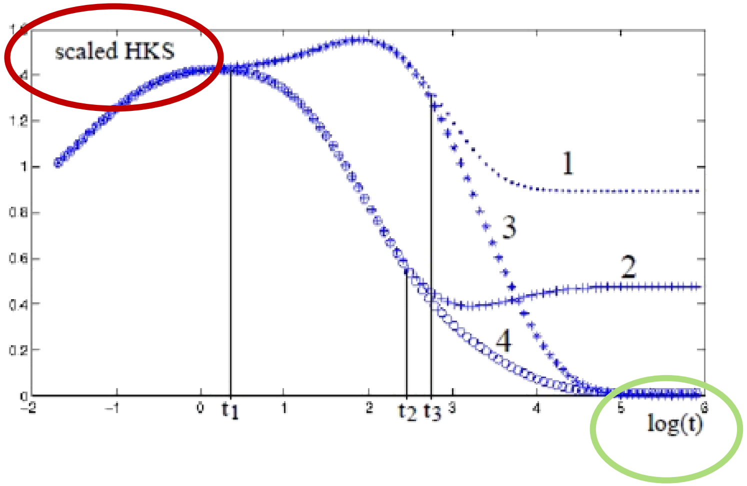


Matching Points

Heuristic adaptions:

→ Scale with $\int_M k_t(x,x)\,dx$ (normalisation)

→ Scale temporal domain logarithmically



Difference between two Heat Kernel Signatures:

$$d_{[t1,t2]}(x,x') = \left(\int_{t_1}^{t_2} \left(\frac{|k_t(x,x) - k_t(x',x')|}{\int_M k_t(x,x)\,dx}\right)^2 d(\log t)\right)^{1/2}$$

Matching Points

$$d_{[t1,t2]}(x,x') = \left(\int_{t_1}^{t_2} \left(\frac{|k_t(x,x) - k_t(x',x')|}{\int_M k_t(x,x)\,dx}\right)^2 d(\log t)\right)^{1/2}$$

Distance function between HKS of purple point to all other points:

Red → difference ≈ 0   Blue → highest difference to purple point

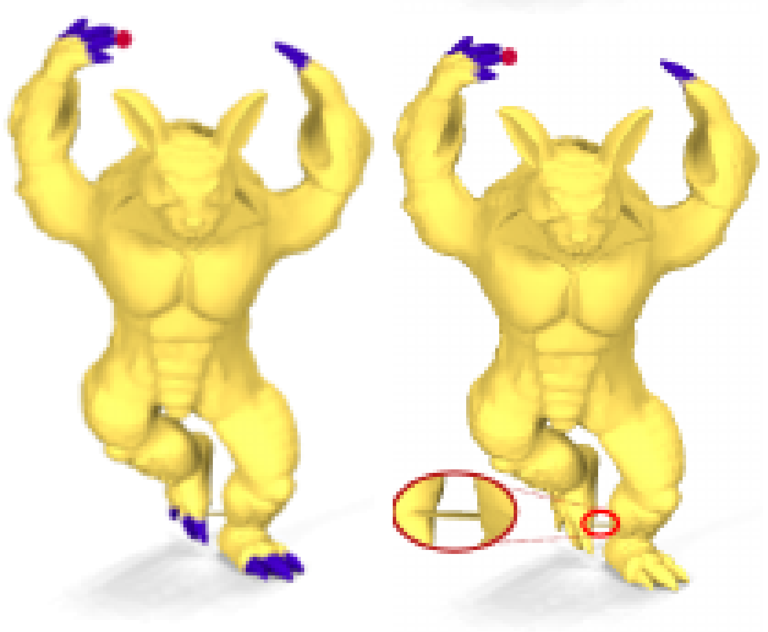
Small t

Larger t

Special Benefits of the HKS

Resilience to Noise:

Small tunnel between feet models noise
Method still works reliably



Properties:
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- ✓ Concise
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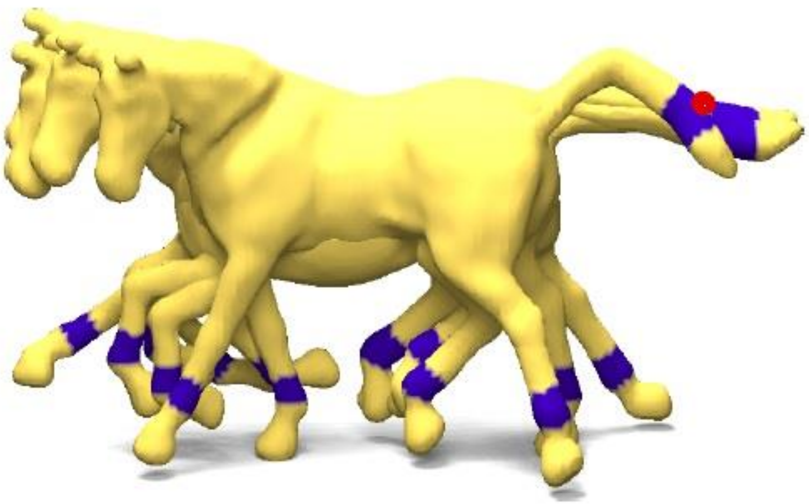
Special Benefits of the HKS

Comparing features across several shapes:
Possible due to invariance under isometric deformations

4 independent datasets (poses) of a horse
Marked: Signatures close to signature of red dot



Properties:
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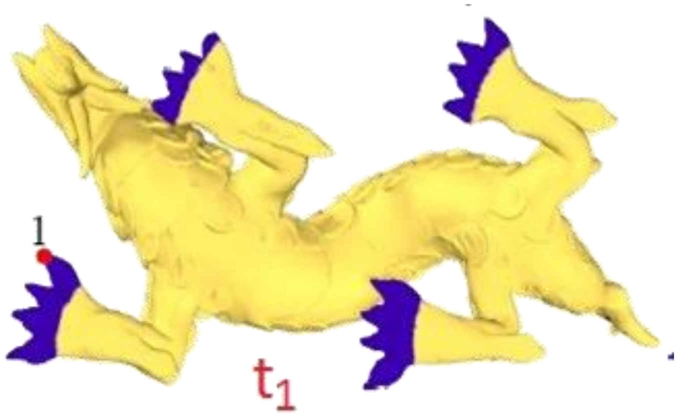
Summary

Effective Shape Matching with the Heat Kernel Signature:
$HKS(x,t) = k_t(x,x)$

Models heat diffusion process on a shape
Only evaluated on the temporal domain
→ Efficient calculation, Concise



Time serves as scaling parameter
→ Multi-Scale

Preserves almost all of the intrinsic information about the geometry
→ Stable, Invariant under isometric deformations

Thank you for your attention