Laplace Beltrami Operator

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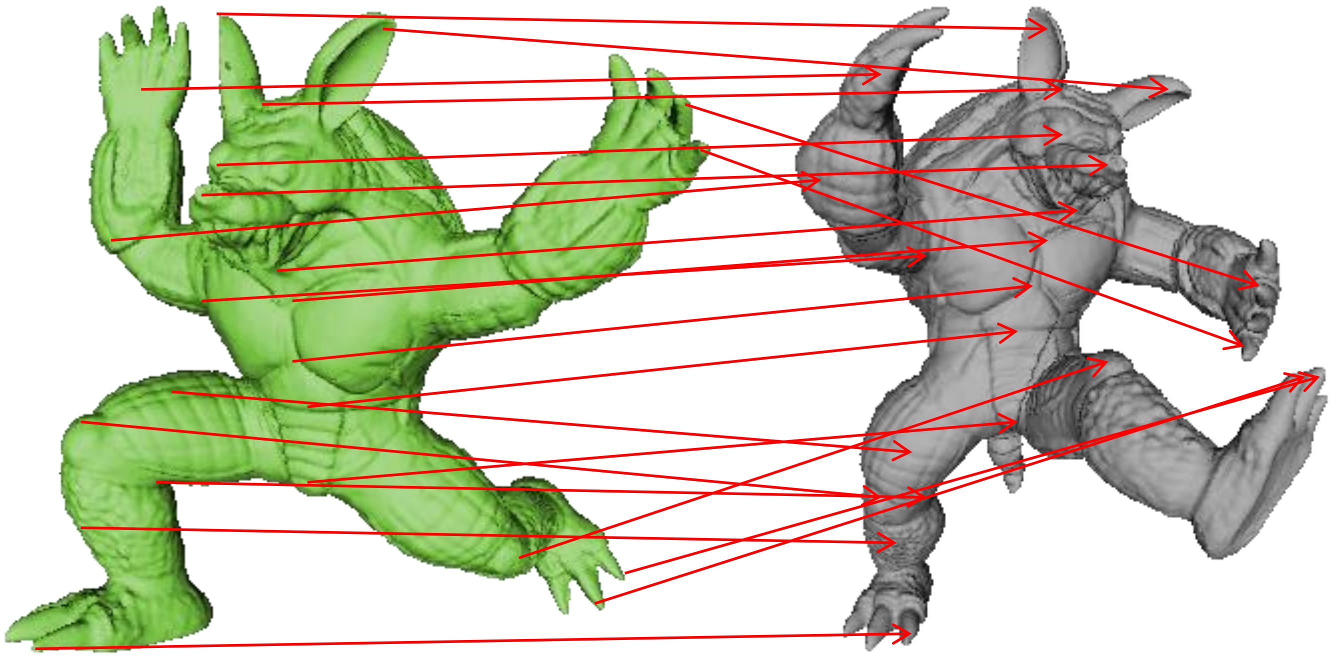
Shape matching



Geodesic distance

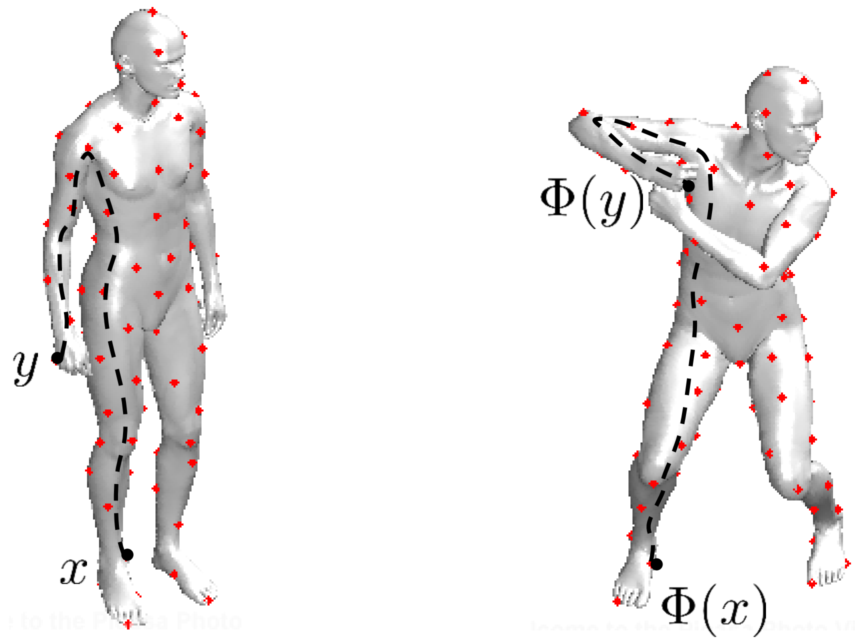
The geodesic distance $d_S(x, y)$ measures the length of the shortest path on the surface S connecting $x \in S$ and $y \in S$.

$$d_S(x, y) = \inf\{L(c) | c : (a, b) \to S,\ c(a) = x,\ c(b) = y\}$$

$d_S : S \times S \to \mathbb{R}^+$ is a metric.

Isometry

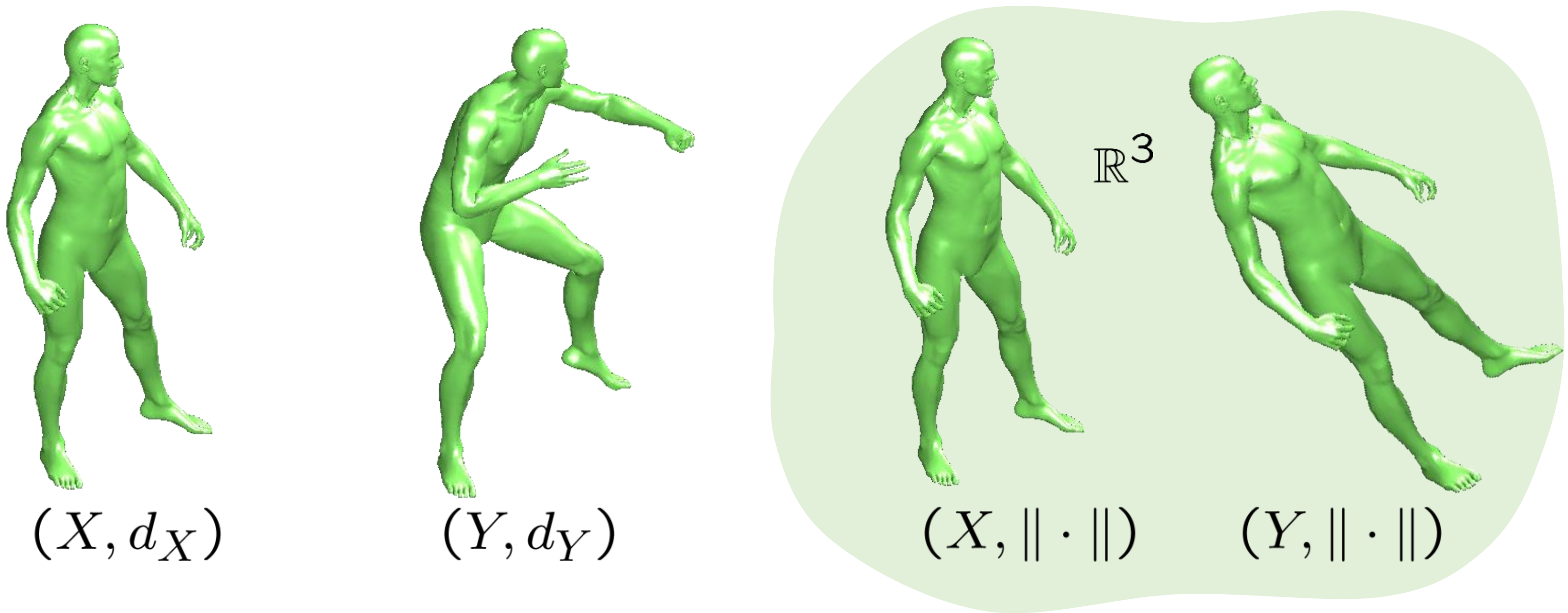
A mapping $\Phi : M \to N$ between two shapes M and N is an isometry if $d_M(x,y) = d_N(\Phi(x), \Phi(y))$ for all points $x, y \in M$. If such a maping exists M and N are called isometric.



$d_M(x,y)$ $\quad\quad$ $d_N(\Phi(x), \Phi(y))$

Many shape matching approaches assume that the shapes to be matched are (nearly) isometric. The task then becomes to find the (almost-)isometry Φ.

Euclidean isometry



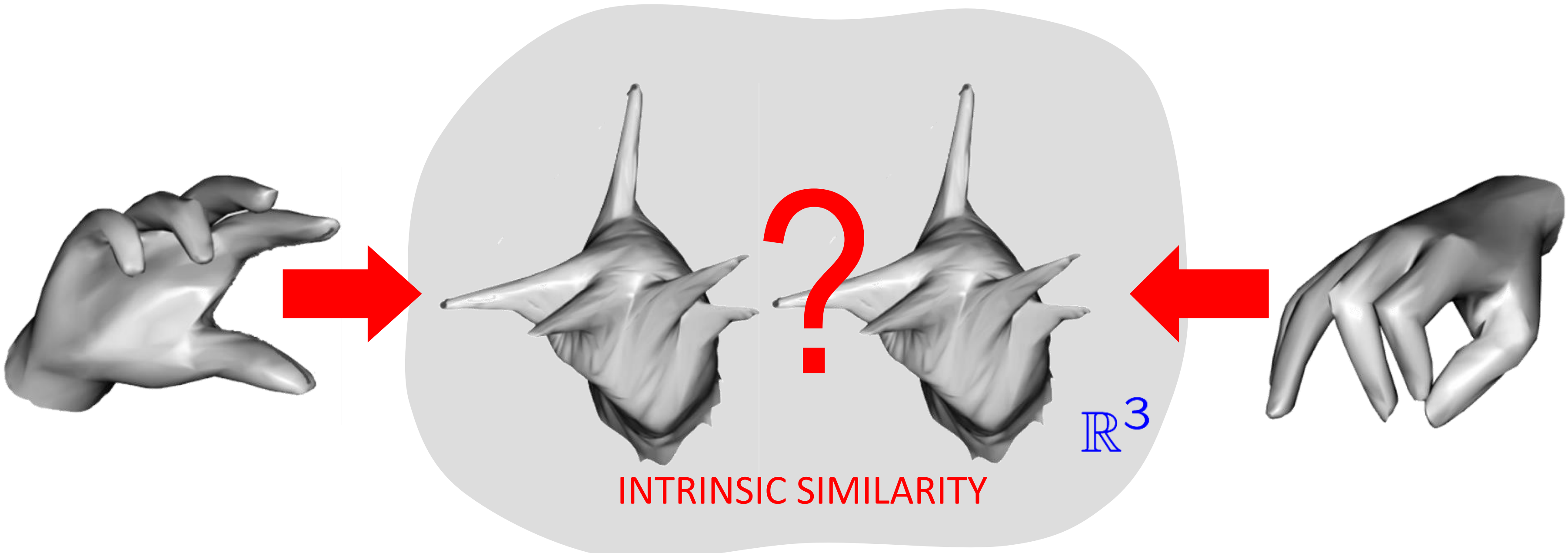
Intrinsic isometry

Two different metric spaces

Euclidean isometry

Part of the same metric space

Distance of canonical forms



Compute canonical forms

EXTRINSIC SIMILARITY OF CANONICAL FORMS

= INTRINSIC SIMILARITY

$$d_{\text{int}}(X, Y) \;=\; d_{\text{ext}}(\varphi(X), \psi(Y))$$

Multidimensional scaling

Find a mapping $\phi : X \to \mathbb{R}^k$ that distorts the distances the least.

Some notation

- Z is an $n \times k$ matrix of canonical form coordinates (each row corresponds to a point): $Z_{j.} = \phi(x_j)$
- $d_{ij}(Z) = \|z_i - z_j\|_{\mathbb{R}^k}$

Distortion/L^2-stress:

$$\sigma_2(Z) = \sum_{i>j} |d_{ij}(Z) - d_X(x_i, x_j)|^2$$

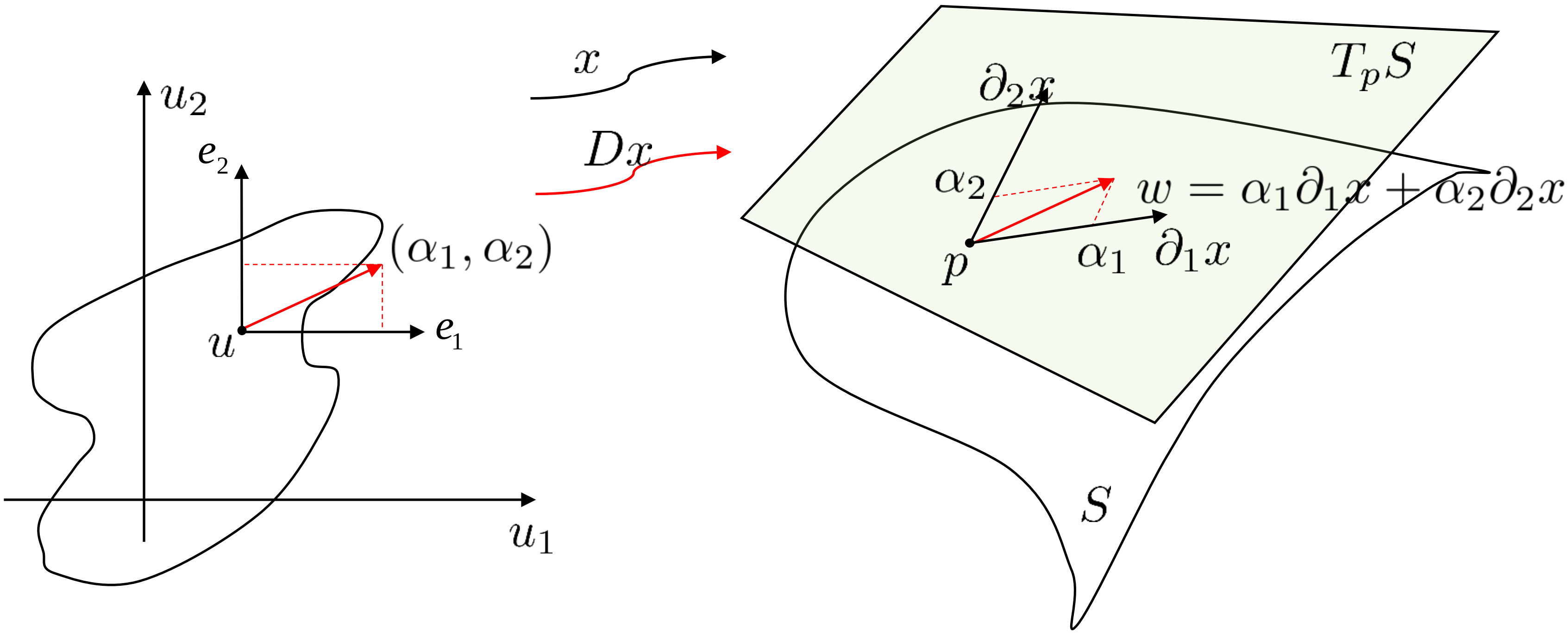
$$= \underbrace{\sum_{i>j} d_{ij}^2(Z)}_{1} - 2\underbrace{\sum_{i>j} d_{ij}(Z) d_X(x_i, x_j)}_{2} + \sum_{i>j} d_X^2(x_i, x_j)$$

Recap integration

Last week we have seen how to integrate scalar functions $f : S \to \mathbb{R}$ defined on a surface:

$$\int_S f(p)dp = \int_U f(x(u)) \cdot \sqrt{\det(\left(Dx\right)^T \left(Dx\right))} du = \int_U f(x(u)) \cdot \sqrt{\det g} du$$

The matrix $g = Dx^T Dx$ is called **first fundamental form** of x.



First fundamental form

Each coordinate map $x : U \to \mathbb{R}$ comes with its own first fundamental form $g = Dx^T Dx$.

Notice that g is in general not constant (as is Dx) but is a (smooth) function $g : U \to \mathbb{R}^{2\times 2}$.

$$g(u) = \begin{pmatrix} g_{11}(u) & g_{12}(u) \\ g_{21}(u) & g_{22}(u) \end{pmatrix} = \begin{pmatrix} \langle \partial_1 x(u), \partial_1 x(u) \rangle & \langle \partial_1 x(u), \partial_2 x(u) \rangle \\ \langle \partial_1 x(u), \partial_2 x(u) \rangle & \langle \partial_2 x(u), \partial_2 x(u) \rangle \end{pmatrix}$$

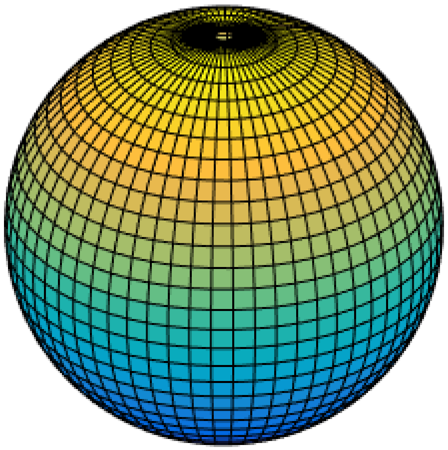
Since Dx has full rank the first fundamental form is a symmetric positive definit.

Sometimes g is also called **Riemannian metric**.

Notation:

$$g^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{\det g} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}$$

Example: Sphere



$$x(u) = (\cos u_1 \cos u_2, \sin u_1 \cos u_2, \sin u_2)^T$$

$$U = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$$

$$Dx = \begin{pmatrix} -\sin u_1 \cos u_2 & -\cos u_1 \sin u_2 \\ \cos u_1 \cos u_2 & -\sin u_1 \sin u_2 \\ 0 & \cos u_2 \end{pmatrix}$$

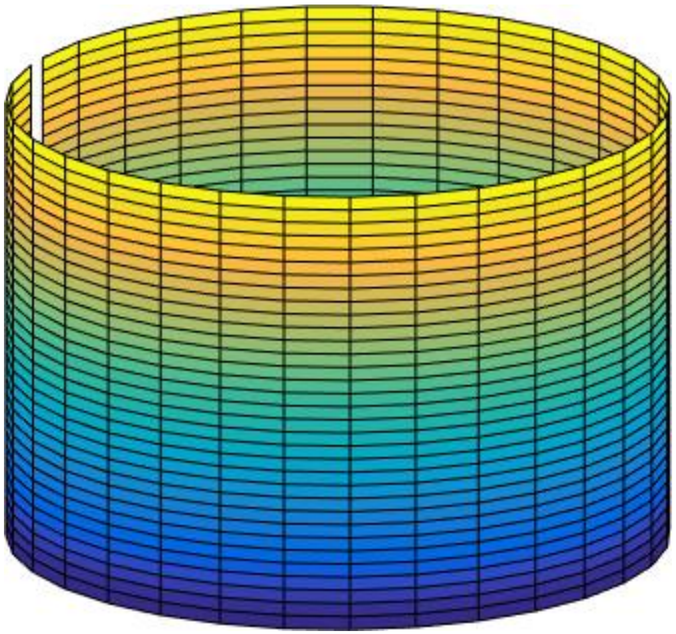
$$g = \begin{pmatrix} \cos^2 u_2 & 0 \\ 0 & 1 \end{pmatrix}$$

If the off-diagonal entries of the first fundamental form vanish, we call x an orthogonal parametrization.

Curves that intersect in a right angle in U also intersect in a right angle on the surface.

This in particular applies to the parameter lines.

Example: Cylinder



$$x(u,v) = (\cos u, \sin u, v)^T$$

$$U = (0, 2\pi) \times \mathbb{R}$$

$$Dx = \begin{pmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & 1 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If the first fundamental form is the identity matrix, we call x an isometric parametrization.
Angles between curves and length of curves in the parameter domain U are preserved under x.

Intrinsics

If M and N are given by coordinate maps (x_j, U_j) and (y_j, U_j) and $\Phi : M \to N$ is an isometry then $g_j^x(x_j^{-1}(p)) = g_j^y(y_j^{-1}(q))$ for all $q = \Phi(p)$.

Thus **intrinsic** quantities are invariant under isometries:

- lenght of curves
- angles between curves
- integrals of functions
- ...

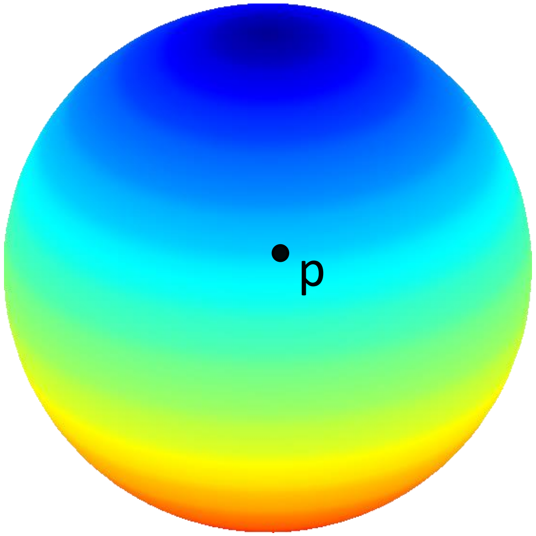
Gradient of a function

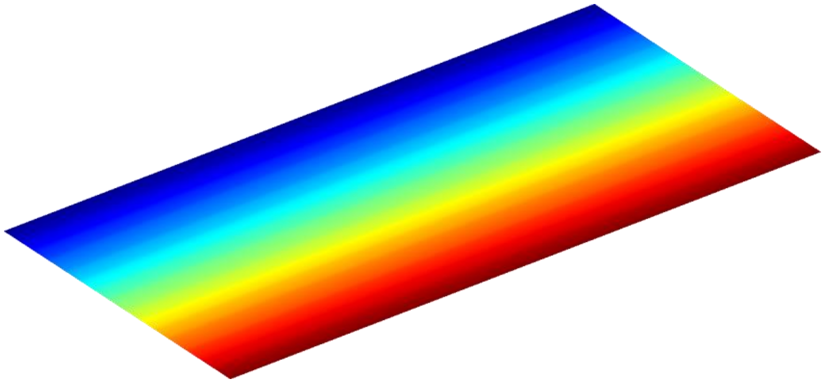
Consider a surface S with parametrization (x, U).

A function $f : S \to \mathbb{R}$ is called differentiable if $\tilde{f} = f \circ x : U \to \mathbb{R}$ is differentiable.

We want to define the **gradient** of f at point $p \in S$.

Goal: Expression in local coordinates.

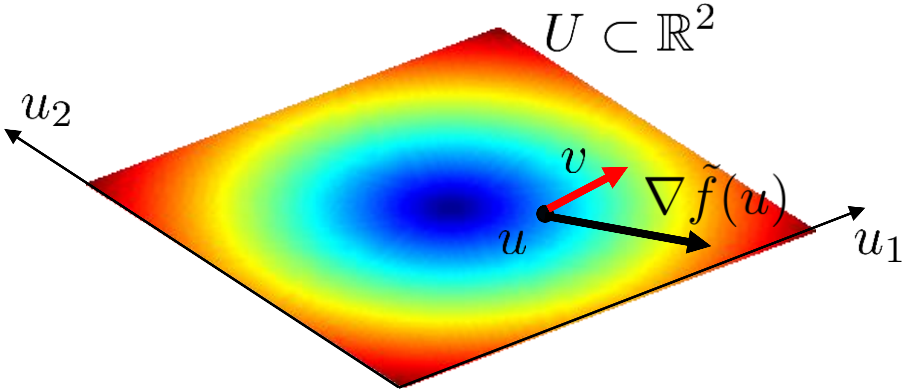




Geometric meaning

Geometric meaning of the gradient:

- the vector that points in the **direction of steepest increase** of $\tilde{f}$
- its length measures the strength of increase
- relationship with the differential of $\tilde{f}$:



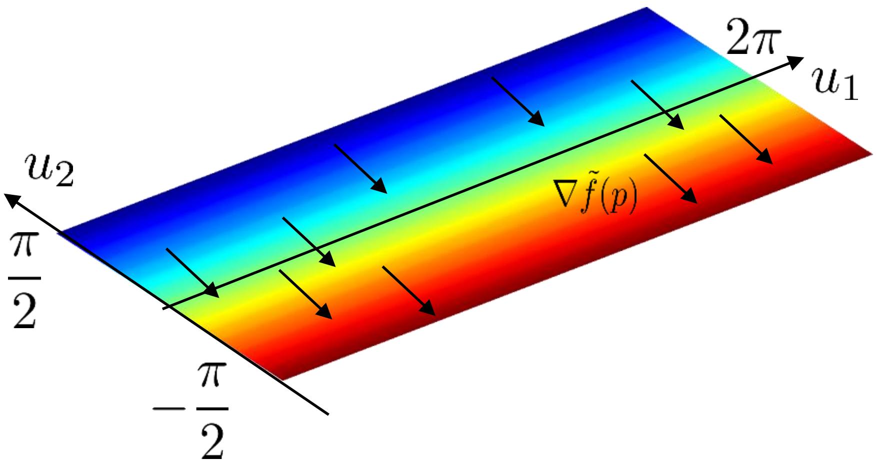
$$d\tilde{f}(u)(\vec{v}) = \lim_{h \to 0} \frac{\tilde{f}(u + h\vec{v}) - \tilde{f}(u)}{h}$$

$$= \frac{d}{dh} \tilde{f}(u + h\vec{v})|_{h=0}$$

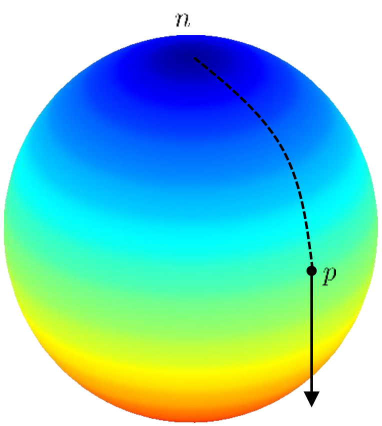
$$= \langle \nabla \tilde{f}(u), \vec{v} \rangle$$

directional derivative of $\tilde{f}$ at u, along direction v

Gradients distance



$$u = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$



$$p = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\nabla f(p) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Gradient

Definition

Let $f : S \to \mathbb{R}$ be a differentiable function. The gradient $\nabla f(p)$ at $p \in S$ is the unique element of T_pS such that

$$\langle \nabla f(p), v \rangle = df(p)[v]$$

(Possible due to Riesz representation theorem)

The gradient in local coordinates

Given $\nabla \tilde{f}$ and g, the coefficients α of $\nabla f = Dx \cdot \alpha \in T_pS$ are given by $\alpha = g^{-1} \nabla \tilde{f}(x^{-1}(p))$

Let β be the coefficients of $v \in T_pS$. Then

$$df(p)[v] = \langle \nabla \tilde{f}(u), \beta \rangle \;\; = \langle \nabla f, v \rangle = \langle \alpha, \beta \rangle_{g(u)}$$

Adjoint operators

Adjoint operator

Let $A : X \to Y$ be a linear and continuous operator between two Hilbertspaces. Then there exists a unique operator $B : Y \to X$ such that

$$\langle Ax, y \rangle_Y = \langle x, By \rangle_X$$

B is again a linear and continuous operator. We call B the adjoint operator of A and write $A^* = B$.

The mapping $x \mapsto \langle Ax, y \rangle_Y \in \mathbb{R}$ is linear and continuous.

Riesz: There exists a unique $z =: By \in X$ such that $\langle Ax, y \rangle_Y = \langle x, z \rangle_X$

Linearity

$$\langle Ax, y_1 + \alpha y_2 \rangle_Y = \langle Ax, y_1 \rangle_Y + \alpha \langle Ax, y_2 \rangle_Y = \langle x, z_1 \rangle_X + \alpha \langle x, z_2 \rangle_X = \langle x, z_1 + \alpha z_2 \rangle_X$$

Adjoint of a matrix

The matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ describes a continuous linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$:

$$A(x) = \mathbf{A}x$$

Adjoint

$$\langle A(x), y\rangle_{\mathbb{R}^m} = \langle \mathbf{A}x, y\rangle_{\mathbb{R}^m} = \langle x, \mathbf{A}^T y\rangle_{\mathbb{R}^n} = \langle x, A^*(y)\rangle_{\mathbb{R}^n}$$

Notice the difference between A and $\mathbf{A}$. In practice A and $\mathbf{A}$ are often identified.
The action of linear operators is often abbriviated:

$$A(x) = Ax$$

Adjoint of gradient

Let $\alpha : U \to \mathbb{R}^2$ be a smooth vectorfield on $U \subset \mathbb{R}^2$ and $\tilde{f} \in C_c^\infty(U)$ a test function.

$$\langle \nabla \tilde{f}, \alpha \rangle = \int_U \alpha_1(u)\partial_1 \tilde{f}(u) + \alpha_2(u)\partial_2 \tilde{f}(u) du$$

$$= -\int_U \partial_1 \alpha_1(u)\tilde{f}(u)du + \int_{\partial U} \alpha_1(s)\tilde{f}(s)\langle e_1, \nu\rangle ds - \int_U \partial_2 \alpha_2(u)\tilde{f}(u)du$$

$$= -\int_U \tilde{f}(u)\,\mathrm{div}\,\alpha(u)du = \langle \tilde{f}, -\mathrm{div}\,\alpha \rangle$$

We say that $-\mathrm{div}$ is *formally* adjoint to ∇.

The gradient is a linear operator but not continuous.

In general one has to carefully choose domain and codomain of operators.

Test function

Test functions

Let again $S \subset \mathbb{R}^n$ be open. The set of testfunctions on S is defined as

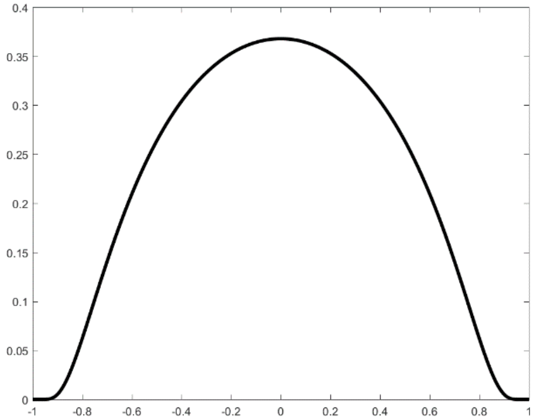
$$C_c^\infty(S) = \{f \in C^\infty(S) | \operatorname{supp} f \subset\subset S\}$$

If S is at the same time compact as are the shapes we consider, then $C_c^\infty(S) = C^\infty(S)$

Example

Consider the function

$$\phi(x) = \begin{cases} \exp(-\frac{1}{1-x^2}) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$



Clearly $\operatorname{supp} \phi = [-1, 1]$ is compact.

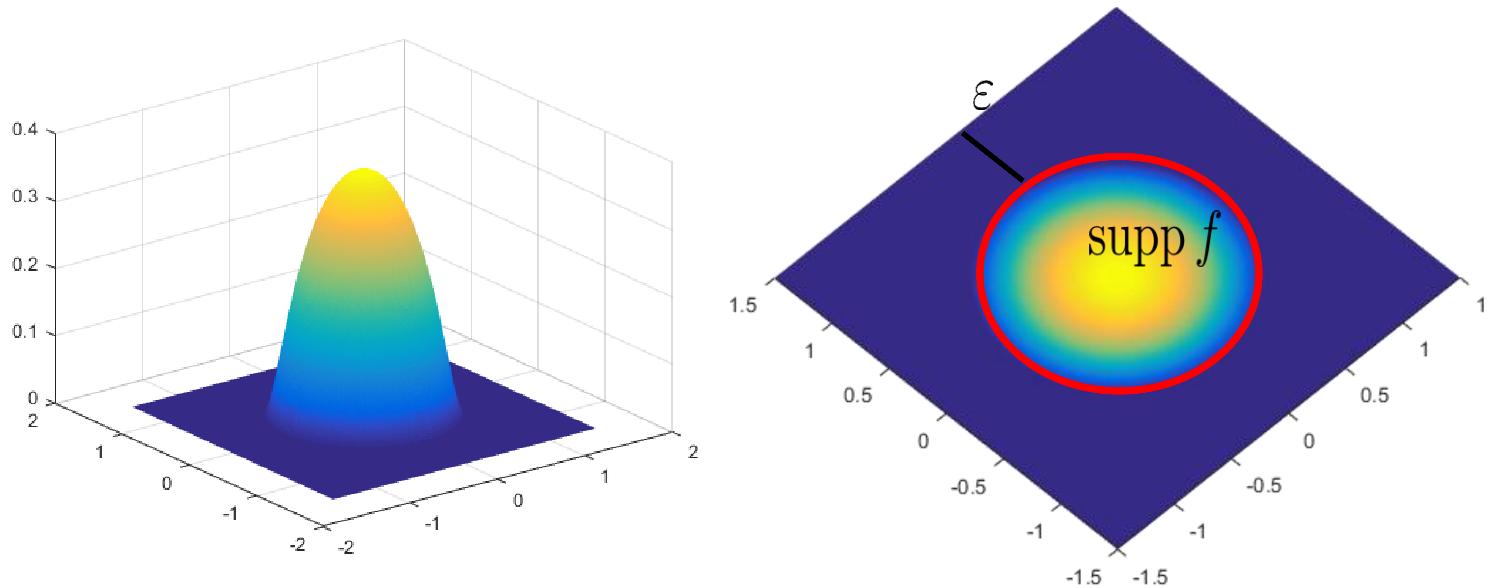
Since $[-1, 1]$ is not contained in the open interval $(-1, 1)$, $\phi \notin C_c^\infty((-1, 1))$ but $\phi \in C_c^\infty(\mathbb{R})$ and $\phi \in C_c^\infty((a, b)) \forall [-1, 1] \subset (a, b)$.

Properties

If the open set $S \subset \mathbb{R}^n$ is bounded and $f \in C_c^\infty(S)$, then there is an $\varepsilon > 0$ such that f vanishes for all points that are closer then ε to the boundary ∂S:

$$\operatorname{dist}(\operatorname{supp} f, \partial S) = \varepsilon > 0$$

As a consequence f and all its derivatives vanish at the boundary of S.



Divergence on manifolds

Smooth vector field

A smooth vectorfield on a compact manifold S is a function

$$V(p) = Dx(x^{-1}(p)) \cdot \begin{pmatrix} \alpha_1(x^{-1}(p)) \\ \alpha_2(x^{-1}(p)) \end{pmatrix}$$

where the coefficient functions $\alpha_i : U \to \mathbb{R}$ are smooth.

Divergence

The divergence of a smooth vectorfield V is the scalar function $\operatorname{div} V : S \to \mathbb{R}$ defined via

$$\int_S \langle \nabla f, V \rangle dp = -\int_S f(p) \operatorname{div} V(p) dp$$

for all test functions $f \in C_c^\infty(S)$.

Laplace Beltrami Operator

Laplace Beltrami Operator (Laplacian)

Let S be a compact manifold and $f \in H^1(S)$ a function on S. We define $\Delta f : S \to \mathbb{R}$ via

$$\int_S \Delta f g dp = -\int_S \langle \nabla f(p), \nabla g(p) \rangle dp$$

for all test functions $g \in C_c^\infty(S)$.

- The Laplacian is the concatenation of divergence and gradient: $\Delta = \mathrm{div} \circ \nabla$
- The Laplacian is a linear operator
- The Laplacian is an intrinsic operator

LBO in local coordinates

We can combine the expressions we have for gradient and divergence and in turn obtain an expression for the Laplacian in local coordinates:

$$\Delta f = \mathrm{div}\nabla f = \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial u_i} \left((g^{-1} \nabla \tilde{f}(u))_i \sqrt{\det g} \right)$$

$$= \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial u_i} \left(\sum_j g^{ij} \frac{\partial \tilde{f}(u)}{\partial u_j} \sqrt{\det g} \right)$$

$$= \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial u_i} \left(g^{ij} \frac{\partial \tilde{f}(u)}{\partial u_j} \sqrt{\det g} \right)$$

For shapes that are (locally) isometric to a subset of $\mathbb{R}^2$ we get

$$\Delta f = \mathrm{div}\nabla f = \frac{\partial^2 f}{\partial x_1 \partial x_1} + \frac{\partial^2 f}{\partial x_2 \partial x_2}$$

Intrinsics

If M and N are given by coordinate maps (x_j, U_j) and (y_j, U_j) and $\Phi : M \to N$ is an isometry then $g_j^x(x_j^{-1}(p)) = g_j^y(y_j^{-1}(q))$ for all $q = \Phi(p)$.

Thus **intrinsic** quantities are invariant under isometries:

- lenght of curves
- angles between curves
- integrals of functions
- gradient operator
- divergence operator
- Laplace Beltrami Operator
- ...

Spectrum of the Laplacian

The Laplacian is a formally self adjoint operator

$$\langle \Delta f, g \rangle = -\langle \nabla f, \nabla g \rangle = \langle f, \Delta g \rangle.$$

As a consequence the eigenvalue problem (**Helmholtz equation**)

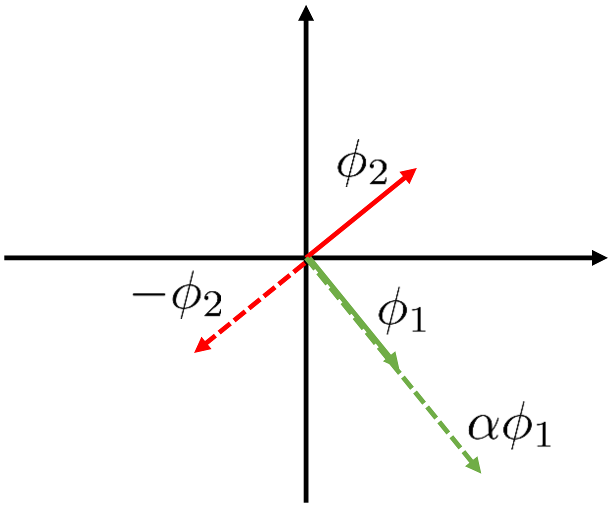
$$\Delta \phi_i = \lambda_i \phi_i$$

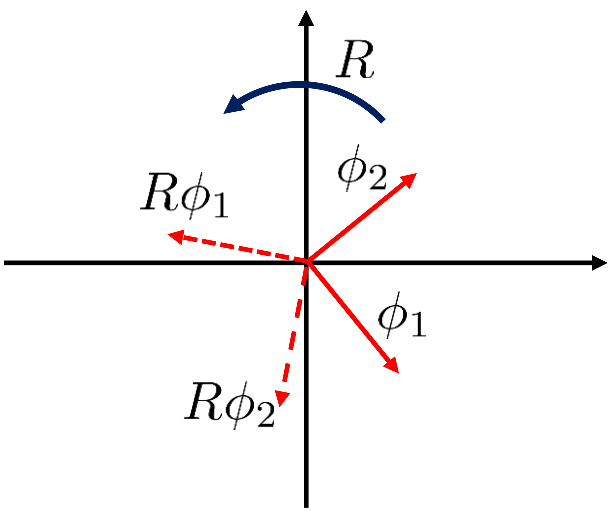
- $\lambda_i \in \mathbb{R}$, in fact we can order them $0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \to -\infty$
- The eigenfunctions can be chosen to be orthogonal: $\langle \phi_i, \phi_j \rangle_{L^2} = \delta_{ij}$

Invariance under isometries

The Laplace Beltrami Operator is an intrinsic operator and therefore invariant to isometric deformations.

However the eigenfunctions are not uniquely defined (signflips, higher dimensional eigenspaces).





Integral of eigenfunctions

For every function $f \in H^1(S)$ we observe

$$\int_S \Delta f = \int_S 1\Delta f = -\int_S \langle \nabla 1, \nabla f \rangle = 0$$

This implies that for every eigenfunction ϕ_i with corresponding eigen-value $\lambda_i \neq 0$:

$$\int_S \phi_i = \frac{1}{\lambda_i} \int_S \Delta \phi_i = 0$$

Dirichlet energy and eigenvalues

Let $\phi_i : S \to \mathbb{R}$ be a (normalized) eigenfunction with corresponding eigenvalue λ_i, and consider the **Dirichlet energy**

$$\int_S \|\nabla \phi_i\|^2 = \int_S \langle \nabla \phi_i, \nabla \phi_i \rangle = -\int_S \phi_i \Delta \phi_i = -\lambda_i \int_S \phi_i \ \phi_i = |\lambda_i|$$

This provides us with a nice characterization of the eigenvalues in terms of the corresponding eigenfunctions.

In particular from the above relation we see that if $\lambda_i = 0$, then ϕ_i must be a constant function. Further $\lambda_i = 0$ is always an eigenvalue of Δ, since $\Delta f = 0$ for any constant function f.

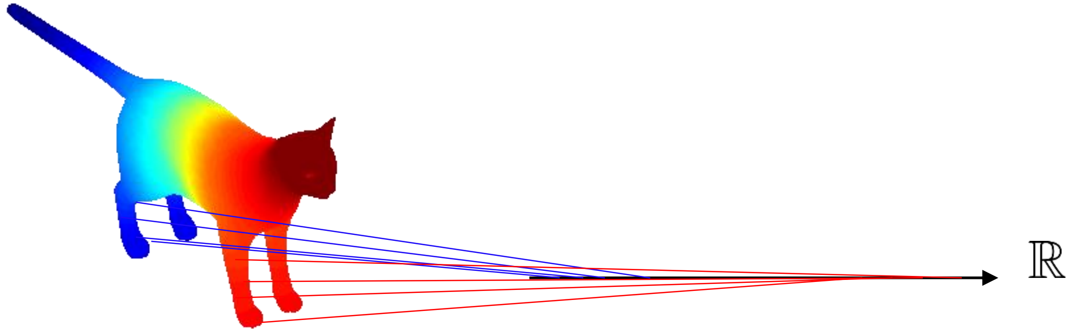
Different euclidean embeddings

Consider the **constrained minimization problem**

$$\min_{\langle f,f\rangle=1} \int_S \|\nabla f\|^2 \quad \text{s.t.} \quad \langle f, \phi_i \rangle = 0 \, \forall 1 \leq i \leq k$$

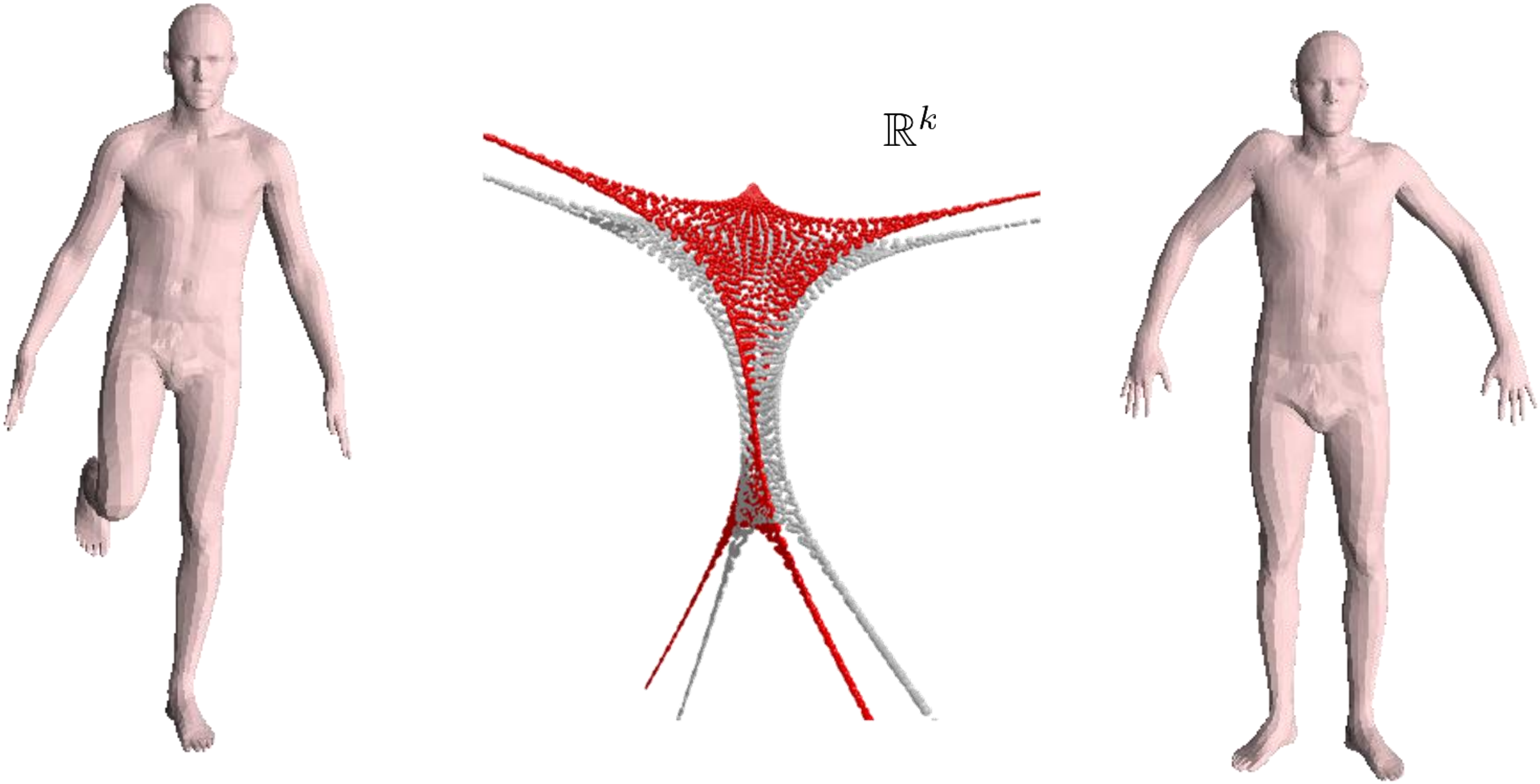
This problem has a very similar structure like the PCA objective. In fact the minimizer is given by $f = \phi_{k+1}$.

Observation: The eigenfunctions are a different way to embed shapes in some $\mathbb{R}^k$. Compared to MDS we are just considering a different stress function.



The first non constant eigenfunction is called **Fiedler vector.**

Spectral embedding

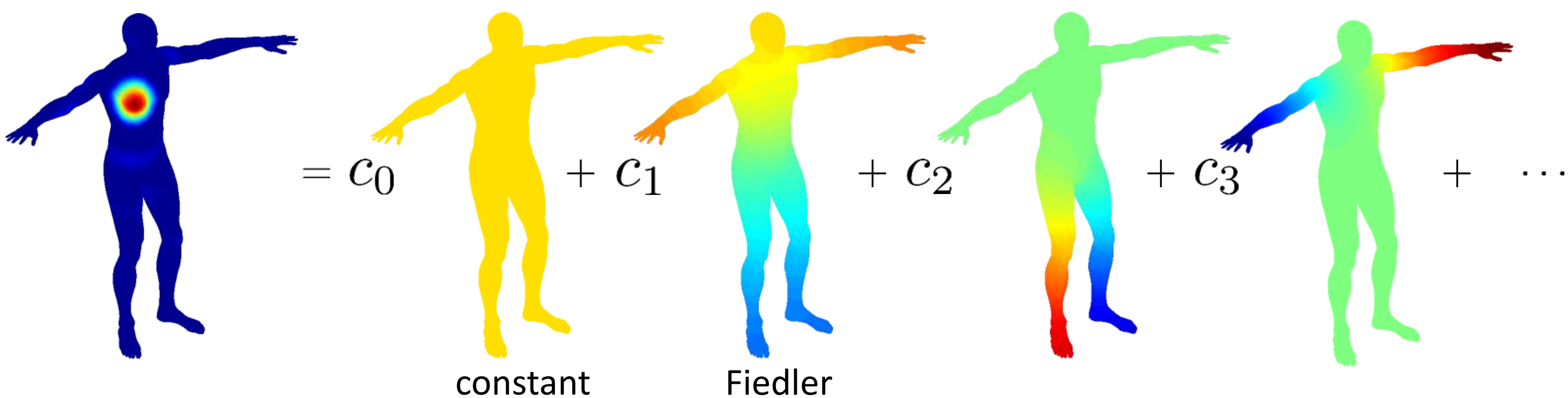


Change of basis

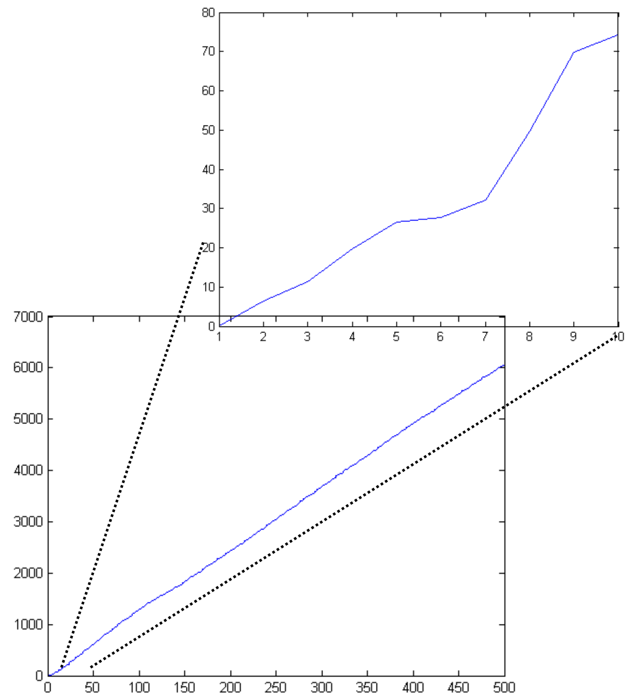
Due to the orthogonality of the eigenfuntions, we can write every function $f \in L^2(S)$ as a linear combination

$$f = \sum_{i=1}^{\infty} c_i \phi_i = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle_{L^2} \phi_i$$

$$\mathbf{f} = \sum_{i=1}^{\infty} \langle \boldsymbol{f}, \boldsymbol{\phi}_i \rangle_{\mathrm{M}} \boldsymbol{\phi}_i$$



Weyl's law



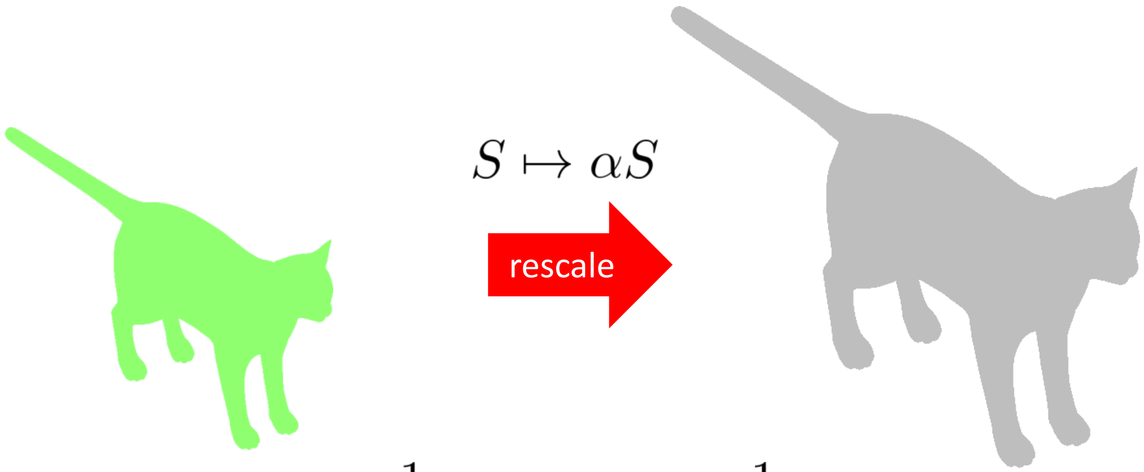
$$|\lambda_j| \sim \frac{\pi}{\int_S da} j \quad \text{for } j \to \infty$$

Influence of scaling

What happens to the eigenvalues and eigenfunctions when we simply rescale a shape?

Weyl's law is already suggesting us that something is going to change.

$$S \mapsto \alpha S$$



$$\phi \mapsto \frac{1}{\alpha}\phi \qquad \lambda \mapsto \frac{1}{\alpha^2}\lambda$$

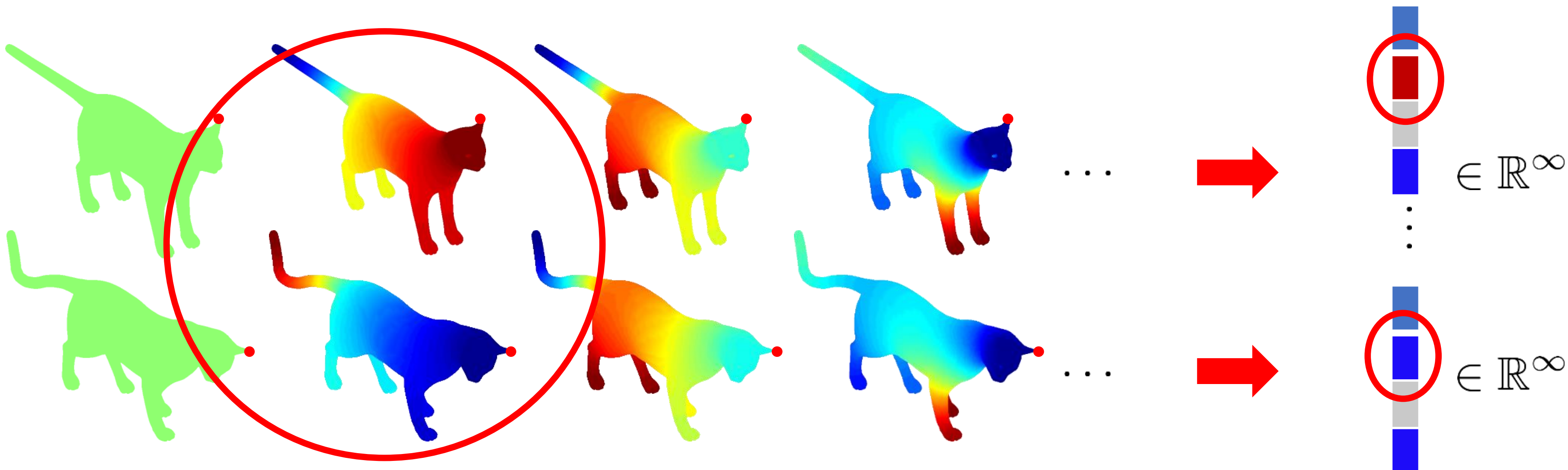
Point features / descriptors

We will consider different descriptors that rely on the LBO.

Global point signature

The most straightforward approach is to map each point $p \in S$ to an infinite-dimensional vector according to the eigenfunctions of the Laplacian:

$$p \mapsto (\varphi_0(p), \varphi_1(p), \varphi_2(p), \dots) \in \mathbb{R}^\infty$$



Global point signature (scale invariant):

$$p \mapsto \left(\frac{\varphi_0(p)}{\sqrt{\lambda_0}}, \frac{\varphi_1(p)}{\sqrt{|\lambda_1|}}, \frac{\varphi_2(p)}{\sqrt{|\lambda_2|}}, \dots \right)$$