Practical Course: Vision-based Navigation Winter Term 2016/2017

Lecture 2: Visual Motion Estimation

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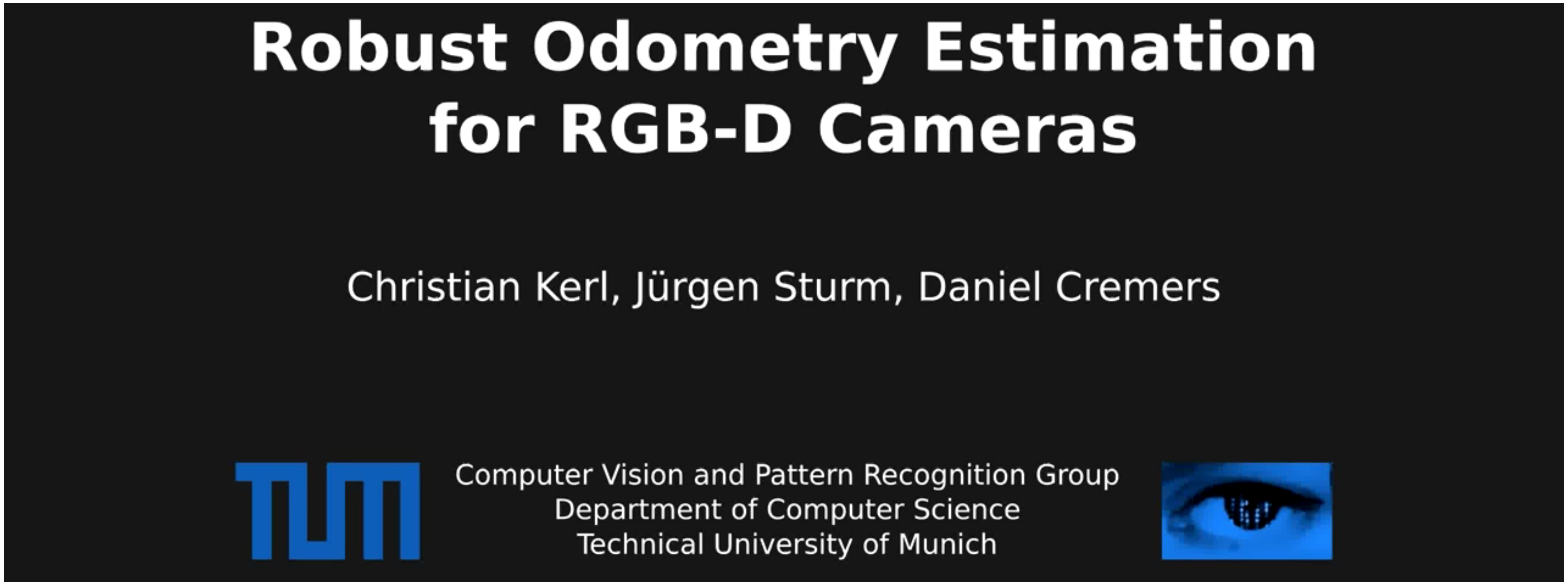
What we will cover today

- Introduction to visual motion estimation approaches
  - Visual odometry (VO) vs. visual SLAM
  - Overview on VO approaches for monocular, stereo, RGB-D cameras
  - The notions of sparse, dense, and direct
- Sparse, keypoint-based visual odometry
- Direct, dense motion estimation
  - Motion representation using the SE(3) Lie algebra
  - Non-linear least squares optimization
  - Direct dense RGB-D odometry

Part 1:
Introduction to Visual Odometry

Visual Motion Estimation a.k.a. Visual Odometry


Robust Odometry Estimation for RGB-D Cameras

Christian Kerl, Jürgen Sturm, Daniel Cremers

Computer Vision and Pattern Recognition Group
Department of Computer Science
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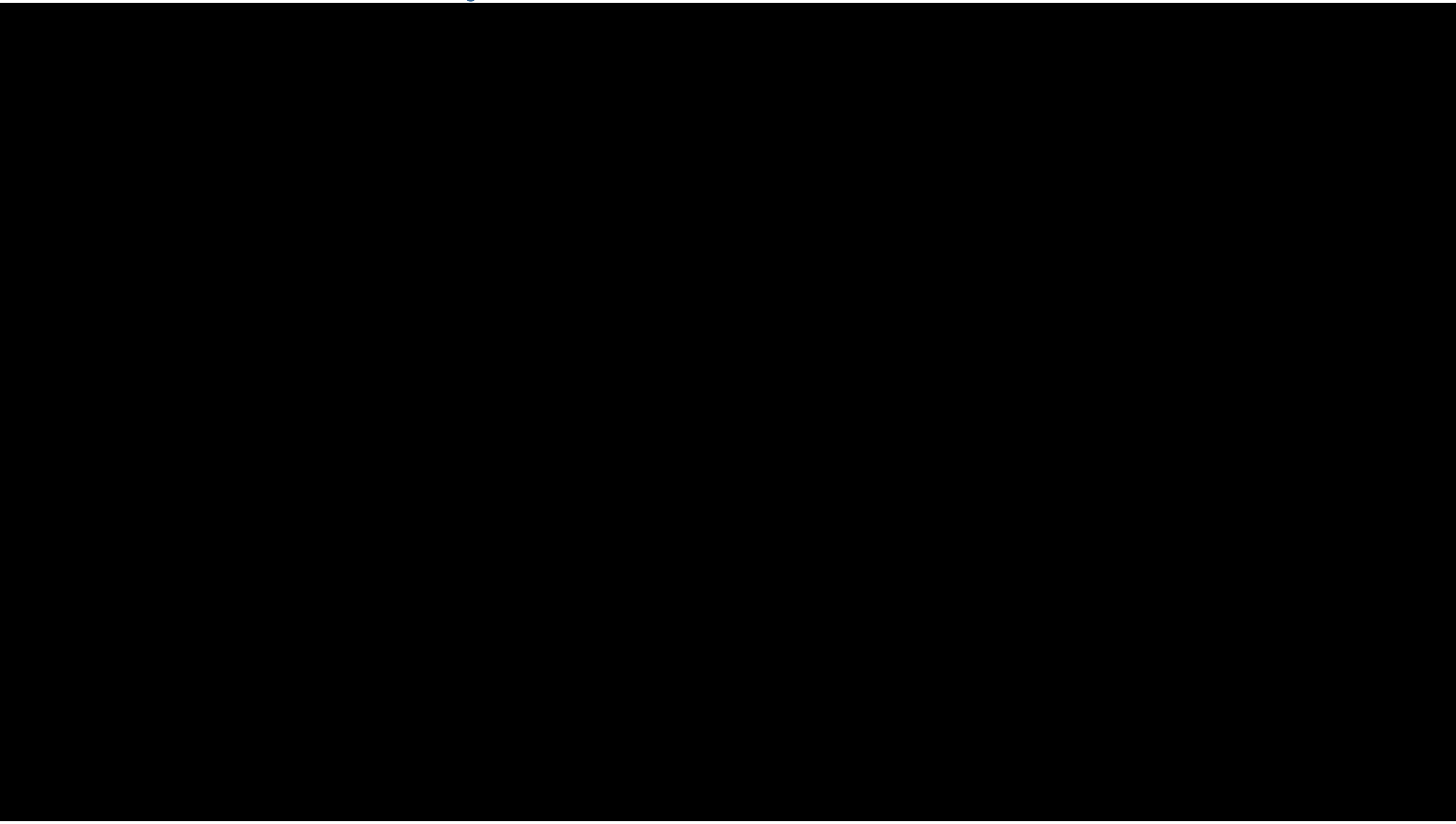
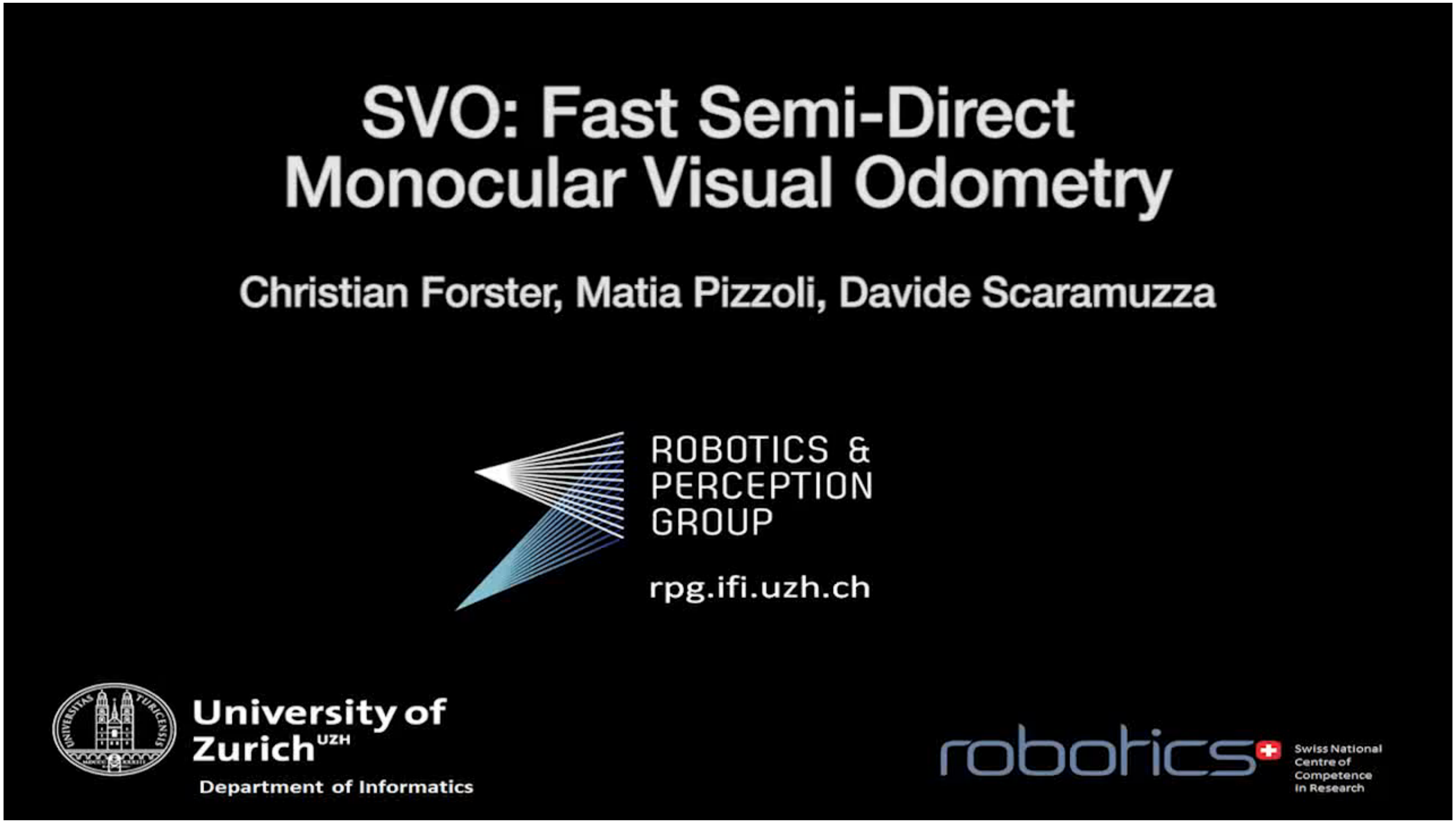
Visual Motion Estimation a.k.a. Visual Odometry



Visual Motion Estimation a.k.a. Visual Odometry


SVO: Fast Semi-Direct Monocular Visual Odometry

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Visual Motion Estimation a.k.a. Visual Odometry

ORB-SLAM

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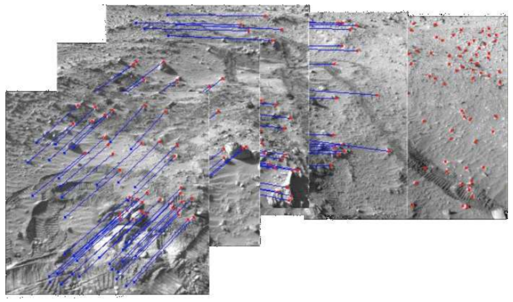
The Term "Visual Odometry"

- Odometry:
  - Greek: „hodos" – path, „metron" – measurement
  - Motion or position estimation from measurements or controls
  - Typical example: wheel encoders

- Visual Odometry (VO):
  - 1980-2004: Dominant research by NASA JPL for Mars exploration rovers (Spirit and Opportunity in 2004)
  - David Nister's „Visual Odometry" paper from 2004 about keypoint-based methods for monocular and stereo cameras



Visual Odometry

- VO is often used to complement other motion sensors
  - GPS
  - Inertial Measurement Units (IMUs)
  - Wheel odometry
  - etc.

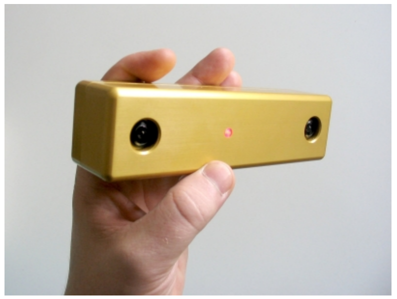
- Important in GPS-denied environments (indoors, underwater, etc.)

- Relation to Visual Simultaneous Localization and Mapping (SLAM):
  - Local (VO) vs. global (VSLAM) consistency
  - VO: 3D reconstruction only at local scale (if at all)
  - VO: Real-time requirements

Sensors for Visual Odometry

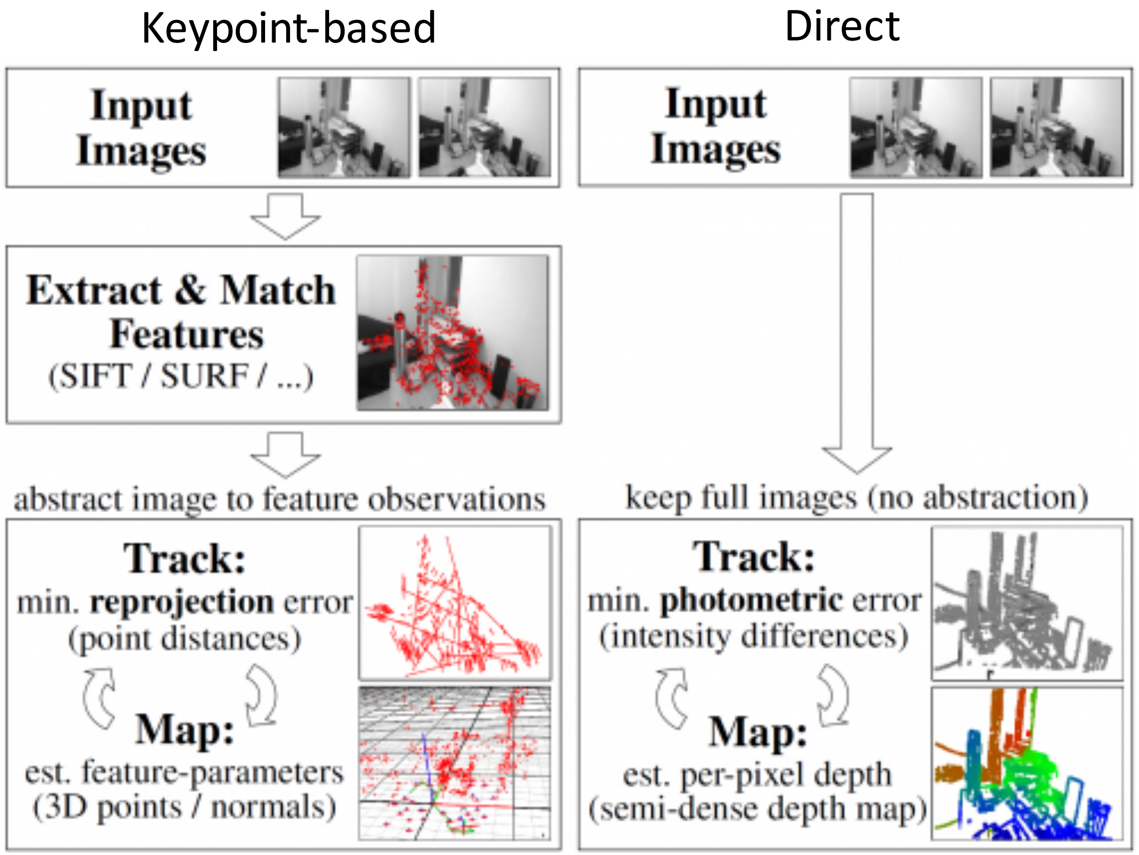
- Monocular:
  - Pros: Low-power, light-weight, low-cost, simple to calibrate and use
  - Cons: requires motion parallax and textured scenes, scale not observable

- Stereo:
  - Pros: depth without motion, less power than active structured light
  - Cons: requires textured scenes, accuracy depends on baseline, requires extrinsic calibration of the cameras, synchronization of the cameras



- Active RGB-D sensors:
  - Pros: also work in untextured scenes, similar to stereo processing
  - Cons: active sensing consumes power, blackbox depth estimation

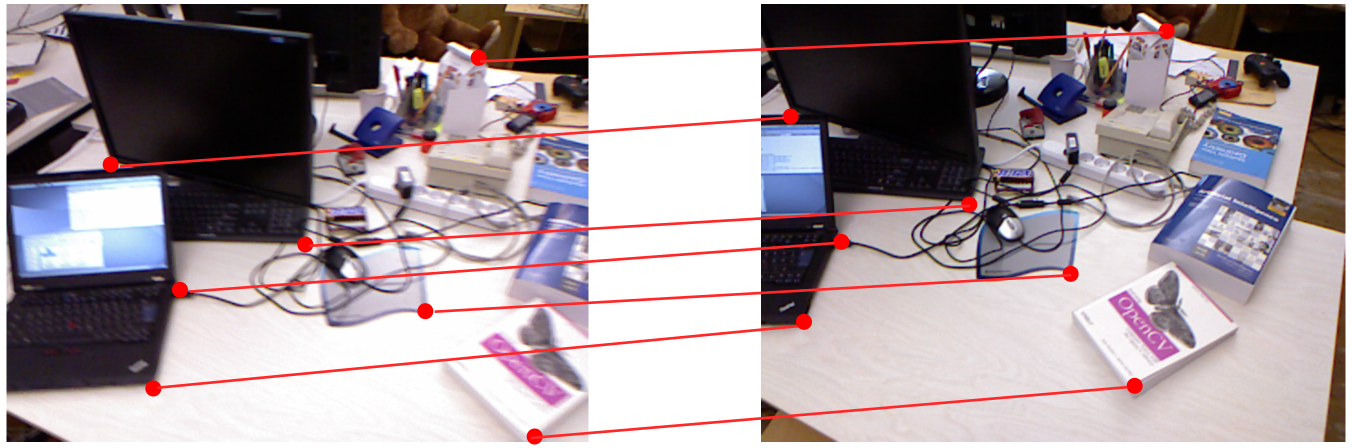
Keypoints, Direct, Sparse, Dense



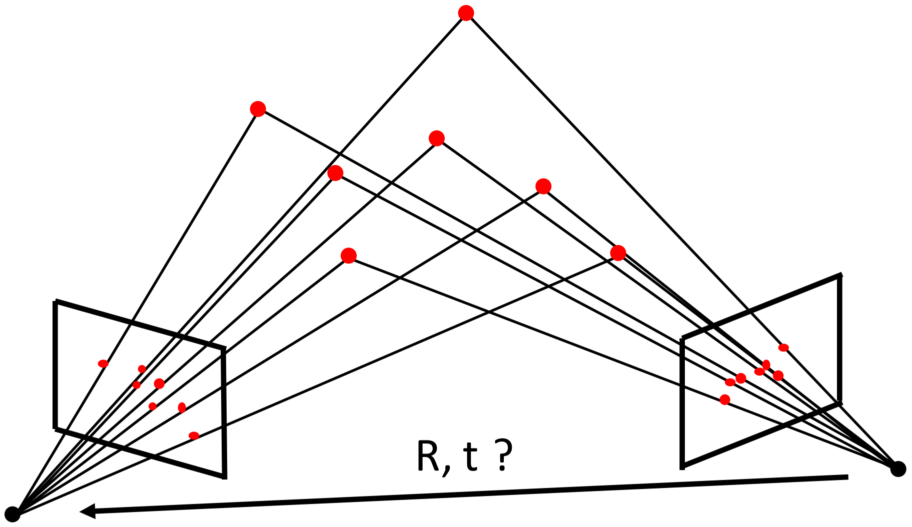
- Sparse: use a small set of selected pixels (keypoints)
- Dense: use all (valid) pixels

Part 2:
Sparse Visual Odometry

Sparse Keypoint-based Visual Odometry



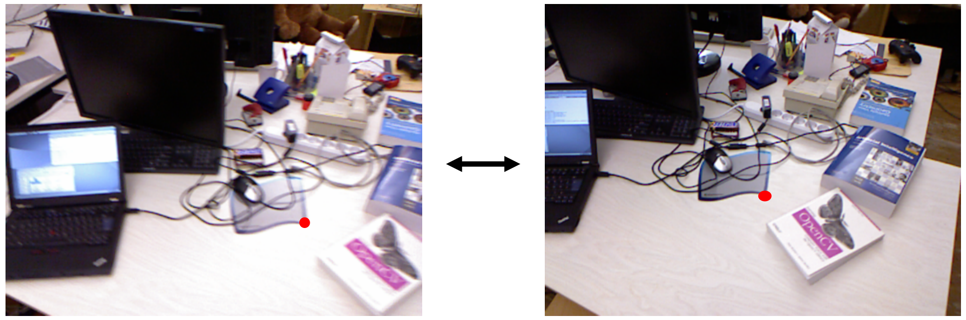
Extract and match keypoints



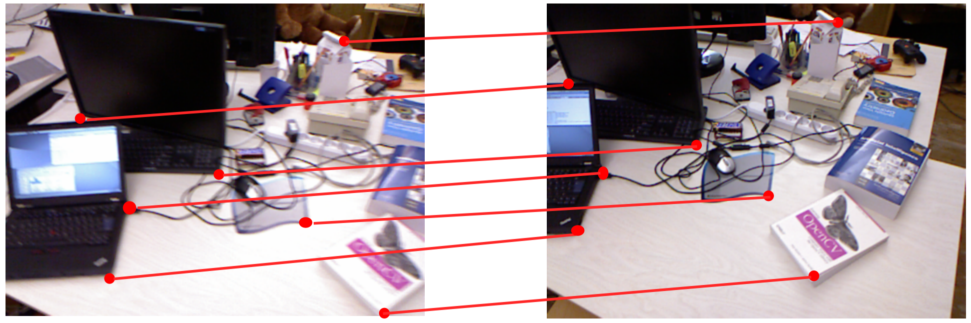
Determine relative camera pose (R, t) from keypoint matches

Keypoint Extraction

- Detection repeatability
  - We want to find the (accurate) image of the same 3D point from different view-points



- Descriptor distinctiveness
  - We want a descriptor that achieves (in the ideal case) a unique and correct association of corresponding keypoints

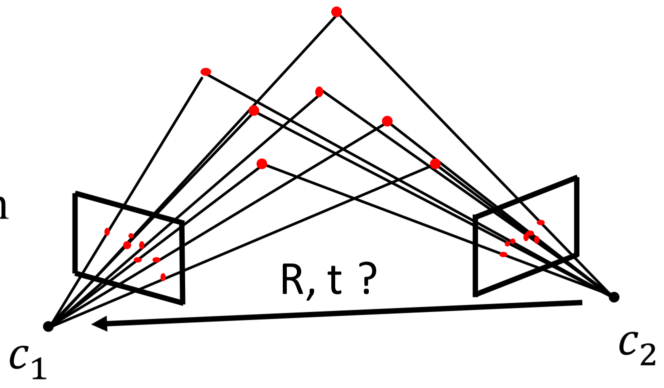


Keypoint Detectors and Descriptors

- Keypoint detection and description in images has been extensively studied
- Nowadays there is plenty of fast and repeatable detectors available, e.g.,
  - Harris corner variants
  - FAST corner variants (e.g. ORB detector)
  - DoG blob variants (SIFT, SURF)
  - Learning-based keypoints
- Many detectors come with a suitable descriptor, e.g.,
  - ORB (binary pixel comparisons locally around keypoint)
  - SIFT/SURF (grayscale gradient patterns locally around keypoint)

Monocular Keypoint-based Motion Estimation

- Monocular case: no depth available at keypoints
- If we knew the relative pose of the cameras and the 3D position of each keypoint match, we could directly compute to which pixels the keypoints should project in each camera image



- To find the unknown pose and 3D positions: minimize the reprojection error of all keypoints (optimization problem)

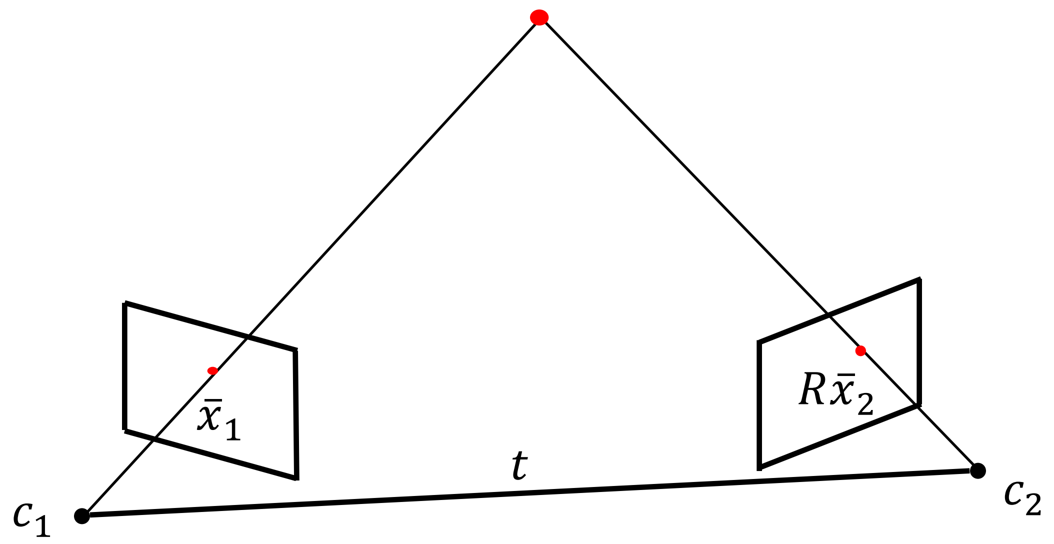
$$E(R, t, x_1, \dots, x_N) = \frac{1}{N} \sum_i \left\| z_{1,i} - \pi(x_i) \right\|_2^2 + \left\| z_{2,i} - \pi(Rx_i + t) \right\|_2^2$$

- Reprojection error: difference between measured and expected pixel position of a keypoint

Uniqueness?
Non-linear projection?

Motion from Epipolar Geometry

- Alternative: examine epipolar geometry more closely



- The rays from each camera to the keypoint and the baseline t are coplanar!

$$\bar{x}_1^T(t \times R\bar{x}_2) = 0 \leftrightarrow \bar{x}_1^T[t]_\times R\bar{x}_2 = 0$$

- The essential matrix $E = [t]_\times R$ captures the relative camera pose
- Each keypoint match provides an „epipolar constraint“
- 8 matches suffice to determine E (8-point algorithm)
- In the uncalibrated case, the camera calibration needs to be subsumed into the so-called fundamental matrix $F = K^{-T} E K^{-1}$

8-Point Algorithm (Longuet-Higgins, 1981)

- Find approximation to essential matrix:
  - Construct matrix $A = (a_1, a_2, \ldots, a_N)^T$ with $a_i = \bar{x}_{1,i} \times \bar{x}_{2,i}$.
  - Apply a singular value decomposition (SVD) on $A = USV^T$ and unstack the 9th column vector of V into $\tilde{E}$
  - Project the approximate $\tilde{E}$ into the (normalized) essential space:
    Determine the SVD of $\tilde{E} = U \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)\, V^T$ and replace the singular values $\sigma_1, \sigma_2, \sigma_3$ with 1,1,0 to find $E = U \operatorname{diag}(1,1,0)\, V^T$
  - Determine one of the following 4 possible solutions that intersect the points in front of both cameras:

$$R = U\, R_Z^T\left(\pm\frac{\pi}{2}\right) V^T$$

$$[t]_\times = U\, R_Z\left(\pm\frac{\pi}{2}\right) \operatorname{diag}(1,1,0) U^T$$

with $$R_Z^T\left(\pm\frac{\pi}{2}\right) = \begin{pmatrix} 0 & \pm 1 & 0 \\ \mp 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

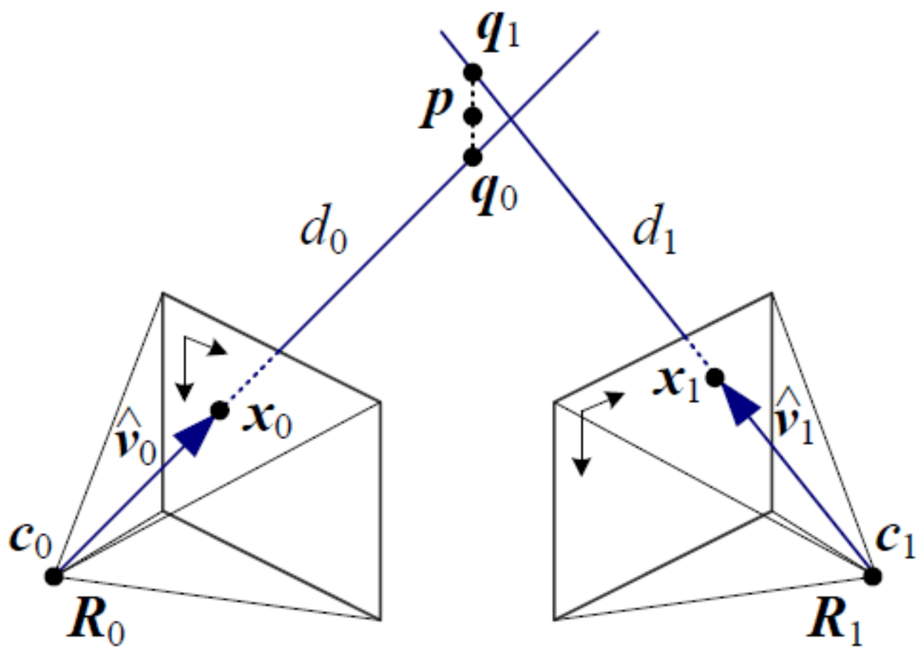
3D Keypoint-based Motion Estimation

- Stereo case: rotation and translation known between the left and right image
- Match keypoints between left and right image, triangulate their 3D positions
- To estimate motion between two stereo image pairs:
  - use 8-point algorithm on keypoints in the left images
  - recover scale from triangulated stereo depth
- Alternatively, since 3D positions of the keypoints known: simpler least-squares optimization of the reprojection error:

$$E(R,t) = \frac{1}{N}\sum_i \left\| z_{1,i} - \pi(R^T x_i - R^T t) \right\|_2^2 + \left\| z_{2,i} - \pi(R x_i + t) \right\|_2^2$$

Triangulation

- **Given:** n cameras $\{M_j = K_j(R_j\ \mathbf{t}_j)\}$
  Point correspondence $\mathbf{x}_0, \mathbf{x}_1$
- **Wanted:** Corresponding 3D point $\mathbf{p}$



Triangulation

- Where do we expect to see $\mathbf{p} = (X\ Y\ Z\ W)^\top$?

$$\hat{x} = \frac{m_{11}X + m_{12}Y + m_{13}Z + m_{14}W}{m_{31}X + m_{32}Y + m_{33}Z + m_{34}W} \qquad \hat{y} = \frac{m_{21}X + m_{22}Y + m_{23}Z + m_{24}W}{m_{31}X + m_{32}Y + m_{33}Z + m_{34}W}$$

- Minimize the residuals

$$\mathbf{p}^* = \arg\min_{\mathbf{p}} \sum_j d(\mathbf{x}_j, \hat{\mathbf{x}}_j)^2$$

Triangulation

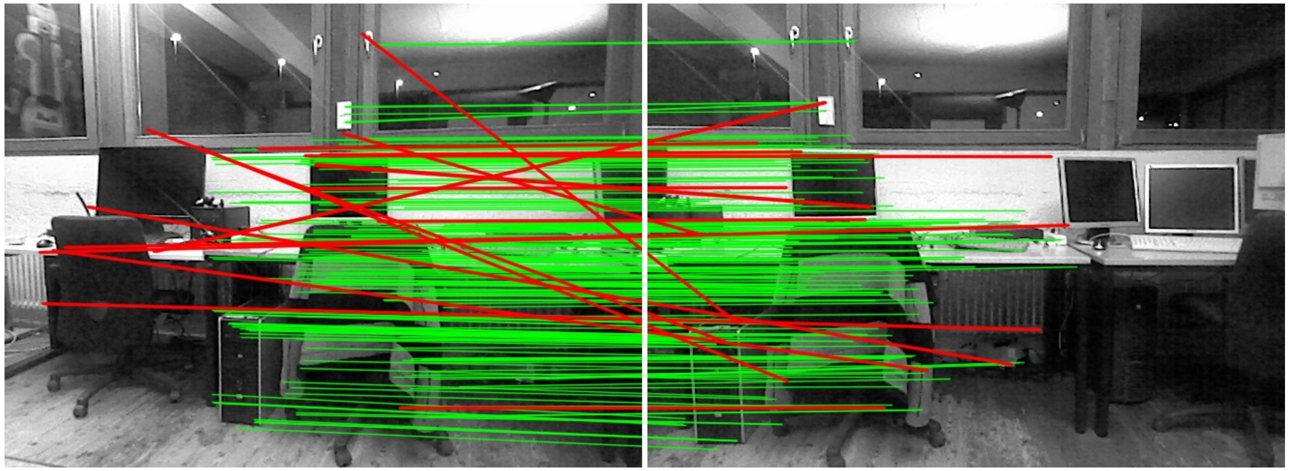
- Multiply with denominator gives

$$0 = (x_j m_{31} - m_{11})X + (x_j m_{32} - m_{12})Y + (x_j m_{33} - m_{13})Z + (x_j m_{34} - m_{14})W$$
$$0 = (y_j m_{31} - m_{21})X + (y_j m_{32} - m_{22})Y + (y_j m_{33} - m_{23})Z + (y_j m_{34} - m_{24})W$$

Solve for $\mathbf{p} = (X\ Y\ Z\ W)^\top$ using:

- Linear least squares with W=1
- Linear least squares using SVD
- Non-linear least squares of the residuals (most accurate)

Robust Keypoint Matching



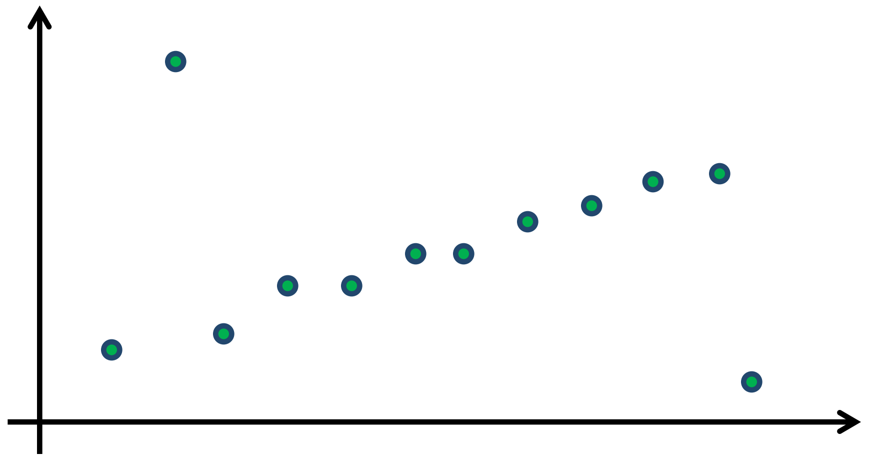
- Keypoint detectors and descriptors not perfect
- Pose estimation very sensitive to wrong correspondences (especially when using the 8-point algorithm)

- Idea: try out different combinations of 8 matches until we find a good fit for most of the overall keypoints

- Random Sample Consensus (RANSAC) algorithm

Robust Estimation

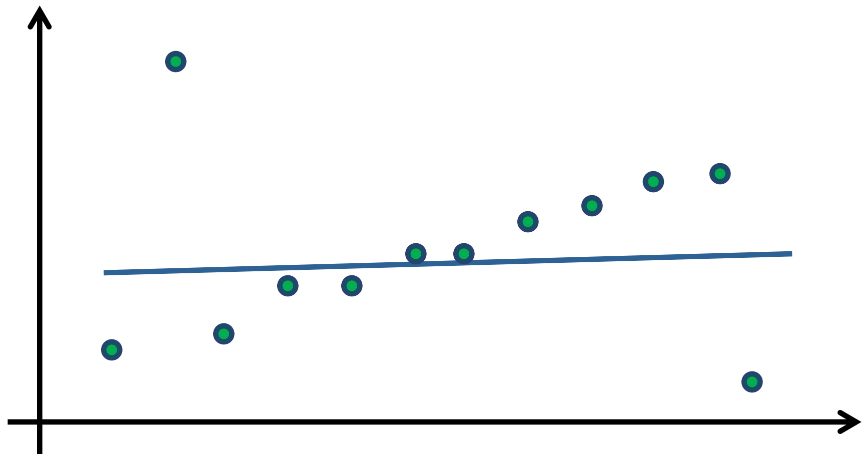
Example: Fit a line to 2D data containing outliers



- Input data is a mixture of
  - Inliers (perturbed by Gaussian noise)
  - Outliers (unknown distribution)
- Let’s fit a line using least squares…

Robust Estimation

Example: Fit a line to 2D data containing outliers



- Input data is a mixture of
  - Inliers (perturbed by Gaussian noise)
  - Outliers (unknown distribution)
- Least squares fit gives poor results!

RANdom SAmple Consensus (RANSAC) [Fischler and Bolles, 1981]

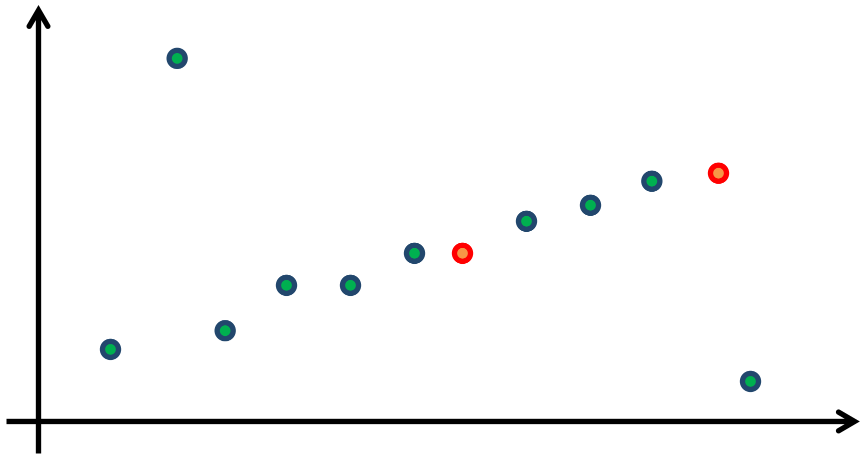
Goal: Robustly fit a model to a data set S which contains outliers

Algorithm:

1. Randomly select a (minimal) subset
2. Instantiate the model from it
3. Using this model, classify all data points as inliers or outliers
4. Repeat 1-3 for N iterations
5. Select the largest inlier set, and re-estimate the model from all points in this set

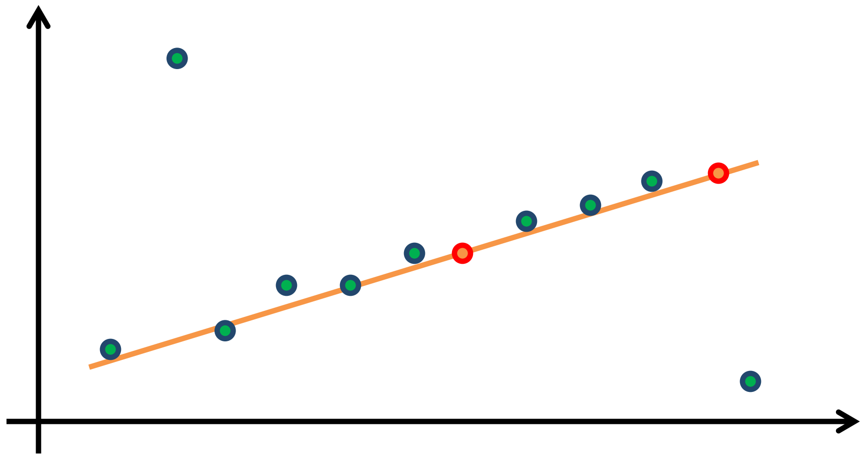
Example

- Step 1: Sample a random subset



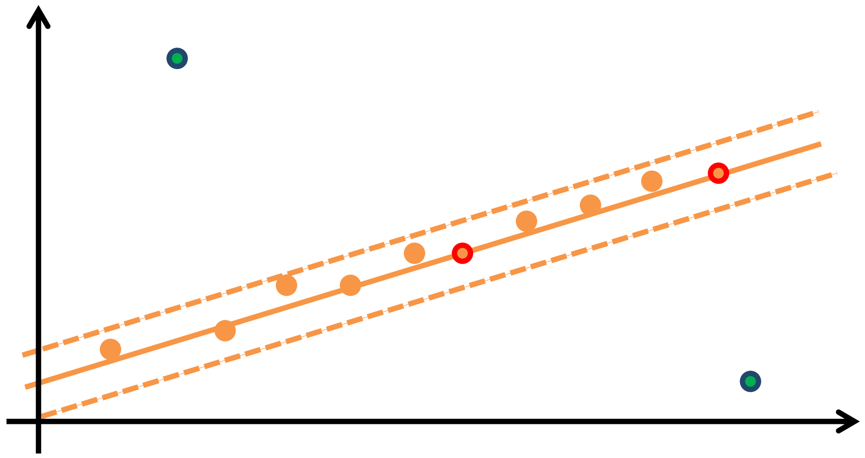
Example

- Step 2: Fit a model to this subset



Example

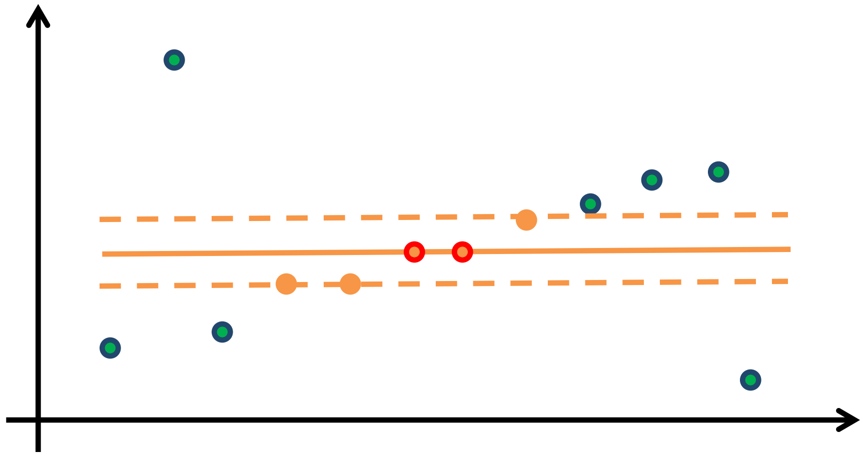
- Step 3: Classify points as inliers and outliers (e.g., using a threshold distance)



→ 10 inliers, 2 outliers

Example

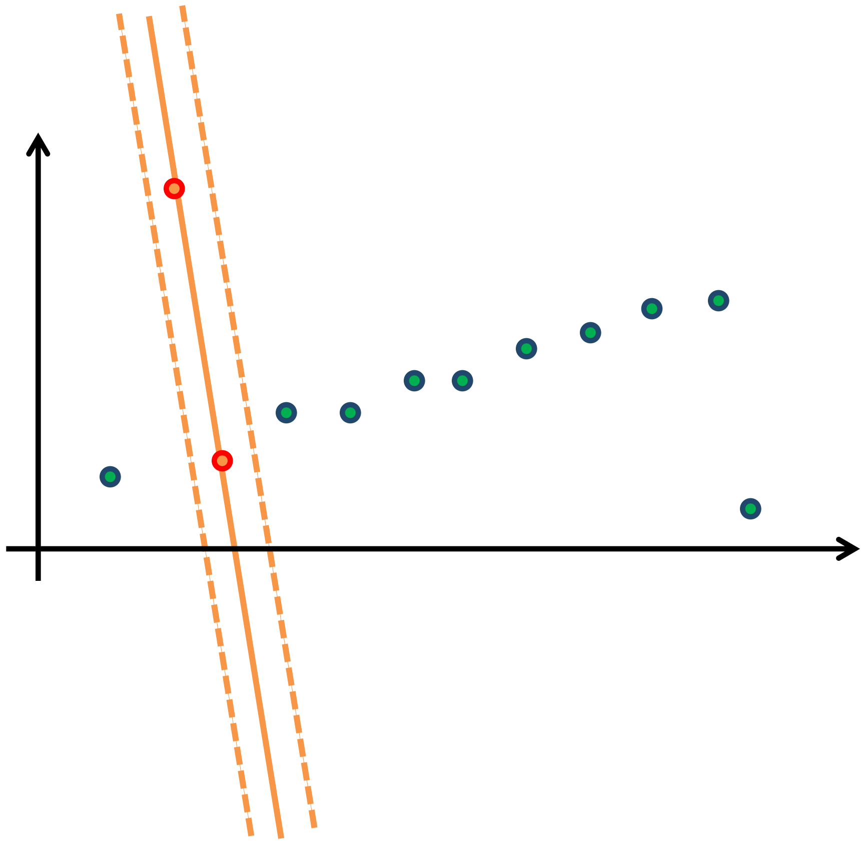
- Step 4: Repeat steps 1-3 for N iterations



Iteration 2:
→ 5 inliers, 7 outliers

Example

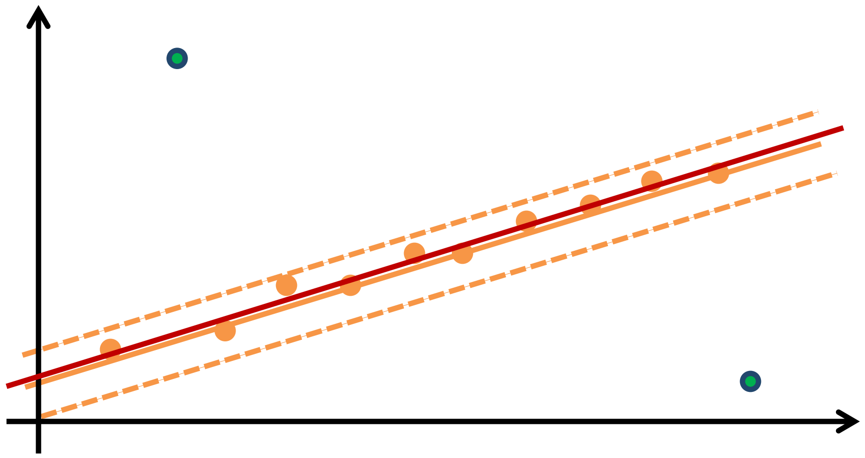
- Step 4: Repeat steps 1-3 for N iterations



Iteration 3:
→ 2 inliers, 10 outliers

Example

- Step 5: Select the best model (most inliers), then re-fit model using all inliers



Best model:
Iteration 1
(10 inliers, 2 outliers)

How Many Iterations Do We Need?

- For a probability of success p, we need

$$N = \frac{\log(1-p)}{\log(1-(1-\epsilon)^s)}$$ iterations

  for subset size s and outlier ratio ϵ
- E.g., for p=0.99:

	Required points s	Outlier ratio ε						
		10 %	20 %	30 %	40 %	50 %	60 %	70 %
Line	2	3	5	7	11	17	27	49
Plane	3	4	7	11	19	35	70	169
Essential matrix	8	9	26	78	272	1177	7025	70188

Summary on RANSAC

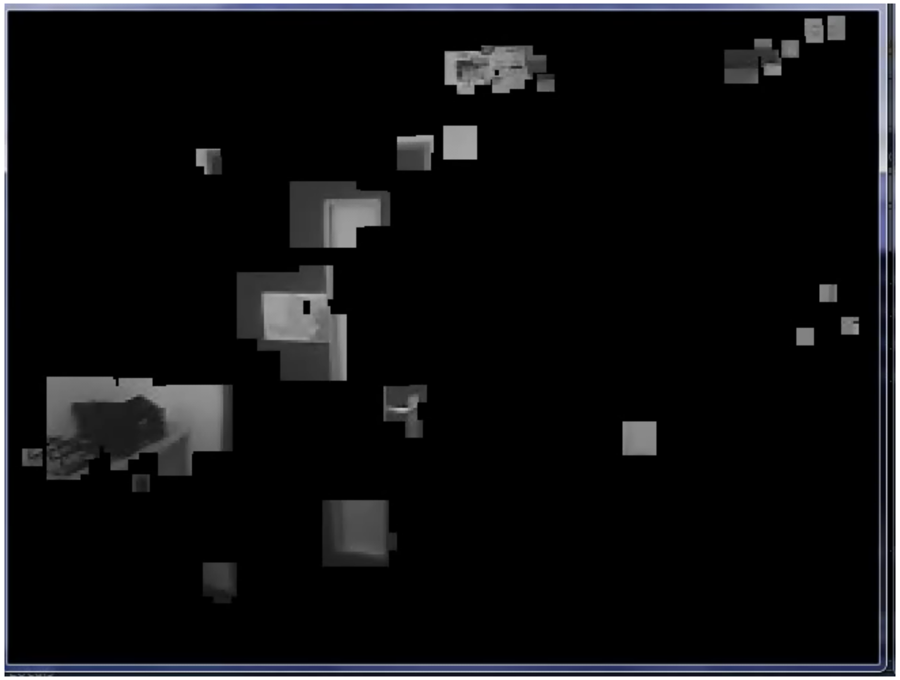
- Efficient algorithm to estimate a model from noisy and outlier-contaminated data
- RANSAC is used today very widely
- Often used in feature matching / visual motion estimation
- Many improvements/variants (e.g., PROSAC, MLESAC, …)

Part 2: Lessons Learned

- How to estimate motion from keypoints from monocular images using the 8-point algorithm
- How to use the 8-point algorithm for stereo and RGB-D
- How to triangulate keypoint matches given the camera pose
- How to separate inliers from outliers using RANSAC

Part 3:
Direct Dense Visual Odometry

Problem with Keypoint-based Methods



Special Euclidean Group SE(3)

- Not all matrices are transformation matrices: Transformation matrices have a special structure

$$\mathbf{T} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \in \mathbf{SE}(3) \subset \mathbb{R}^{4\times 4}$$

  - Translation $\mathbf{t}$ has 3 degrees of freedom
  - Rotation $\mathbf{R}$ has 3 degrees of freedom
- They form a group which we call SE(3). The group operator is matrix multiplication:

$$\cdot : \mathbf{SE}(3) \times \mathbf{SE}(3) \to \mathbf{SE}(3)$$
$$\mathbf{T}_B^A \cdot \mathbf{T}_C^B \mapsto \mathbf{T}_C^A$$

- The operator is associative, but not commutative!
- There is also an inverse and a neutral element

Parametrizations of SE(3)

- Translation $\mathbf{t}$ has 3 degrees of freedom
- Rotation $\mathbf{R}$ has 3 degrees of freedom

$$\mathbf{T} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \in \mathbf{SE}(3) \subset \mathbb{R}^{4\times 4}$$

- Different parametrizations θ of $\mathbf{T}(\theta)$
  - Direct matrix representation
  - Quaternion / translation
  - Axis,angle / translation
  - Later: Twist coordinates in Lie Algebra se(3) of SE(3)

Pose Parametrization for Optimization

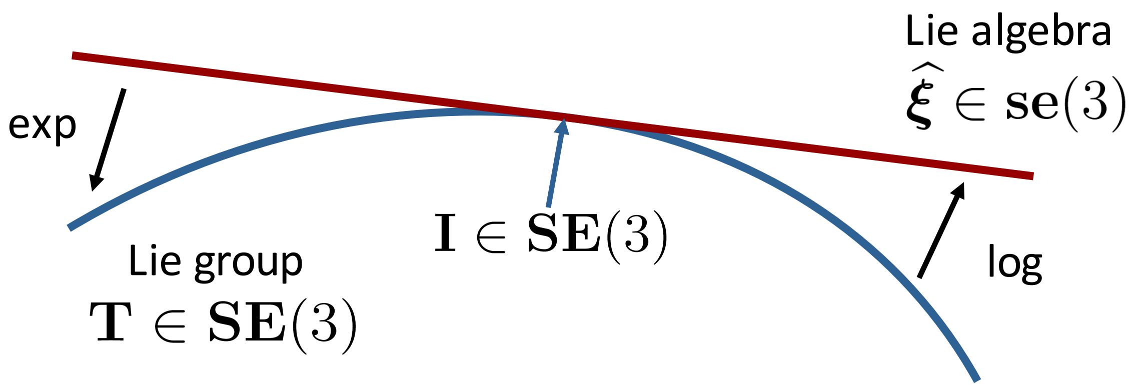
- Let's say we want to optimize a cost function $E(\theta)$ for the pose in some θ parametrization
- We need to set $\nabla_\theta E(\theta) = 0$

  which we can tackle using gradient descent (or higher-order methods) by making steps on θ

$$\theta \leftarrow \theta - \lambda \nabla_\theta E(\theta)$$

- When we determine the derivative of $E(\theta)$, we will require the derivative of $\mathbf{T}(\theta)$ for θ, which should have no singularities
- We also update the pose parametrization, which requires a minimal representation

SE(3) Lie Algebra for Representing Motion



Lie algebra
$\widehat{\boldsymbol{\xi}} \in \mathbf{se}(3)$
exp
$\mathbf{I} \in \mathbf{SE}(3)$
log
Lie group
$\mathbf{T} \in \mathbf{SE}(3)$

$$\boldsymbol{\xi} := \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} \in \mathbb{R}^6$$

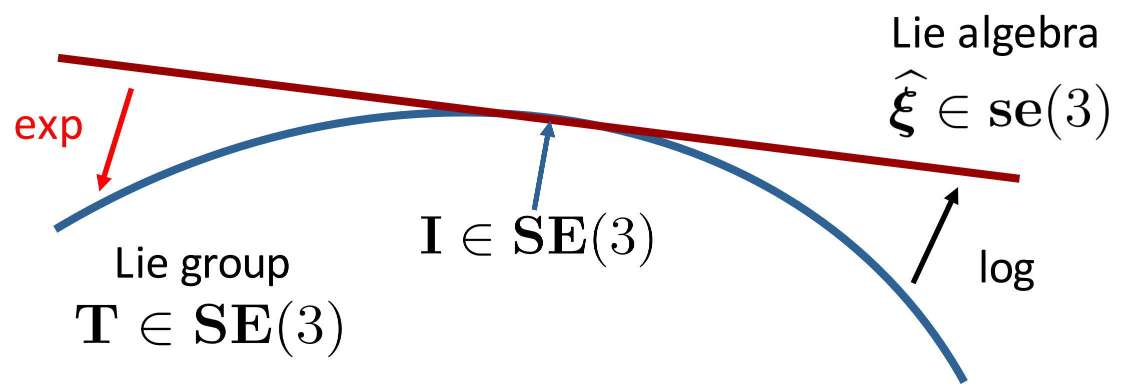
$$\boldsymbol{\omega} \in \mathbb{R}^3 \qquad \widehat{\boldsymbol{\omega}} := [\boldsymbol{\omega}]_\times$$

$$\mathbf{v} \in \mathbb{R}^3$$

$$\widehat{\boldsymbol{\xi}} := \begin{pmatrix} \widehat{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{pmatrix} \in \mathbb{R}^{4\times 4}$$

- SE(3) is also a smooth manifold which makes it a Lie group
- The SE(3) Lie Algebra se(3) provides an elegant way to parametrize poses for optimization
- Its elements $\widehat{\boldsymbol{\xi}} \in \mathbf{se}(3)$ rm the tangent space of SE(3) at its identity $\mathbf{I} \in \mathbf{SE}(3)$
- The se(3) elements can be interpreted as rotational and translational velocities applied for some duration (twist) that explain the infinitesimal motion away from the identity transformation

Exponential Map of SE(3)

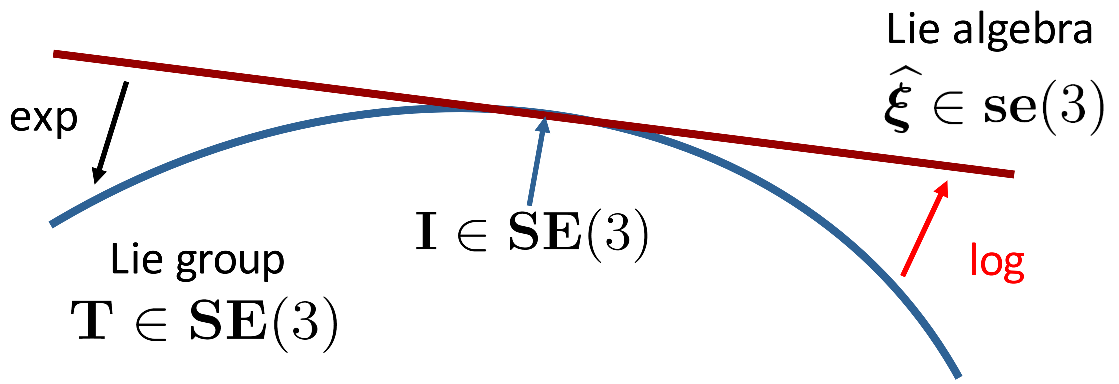

Lie algebra
$\widehat{\boldsymbol{\xi}} \in \mathbf{se}(3)$
exp
$\mathbf{I} \in \mathbf{SE}(3)$
log
Lie group
$\mathbf{T} \in \mathbf{SE}(3)$

- The exponential map finds the transformation matrix for a twist:

$$\exp\left(\widehat{\boldsymbol{\xi}}\right) = \begin{pmatrix} \exp\left(\widehat{\boldsymbol{\omega}}\right) & \mathbf{A}\mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix}$$

$$\exp\left(\widehat{\boldsymbol{\omega}}\right) = \mathbf{I} + \frac{\sin|\omega|}{|\omega|}\widehat{\boldsymbol{\omega}} + \frac{1-\cos|\omega|}{|\omega|^2}\widehat{\boldsymbol{\omega}}^2 \qquad \mathbf{A} = \mathbf{I} + \frac{1-\cos|\omega|}{|\omega|^2}\widehat{\boldsymbol{\omega}} + \frac{|\omega| - \sin|\omega|}{|\omega|^3}\widehat{\boldsymbol{\omega}}^2$$

Logarithm Map of SE(3)



Lie algebra
$\widehat{\boldsymbol{\xi}} \in \mathbf{se}(3)$
exp
$\mathbf{I} \in \mathbf{SE}(3)$
log
Lie group
$\mathbf{T} \in \mathbf{SE}(3)$

- The logarithm maps twists to transformation matrices:

$$\log(\mathbf{T}) = \begin{pmatrix} \log(\mathbf{R}) & \mathbf{A}^{-1}\mathbf{t} \\ \mathbf{0} & 0 \end{pmatrix}$$

$$|\omega| = \cos^{-1}\left(\frac{\operatorname{tr}(\mathbf{R}) - 1}{2}\right) \qquad \log(\mathbf{R}) = \frac{|\omega|}{2\sin|\omega|}\left(\mathbf{R} - \mathbf{R}^T\right)$$

Optimization with Twist Coordinates

- How are twists useful in optimization?
- They provide a minimal representation without singularities close to identity
- Since SE(3) is a smooth manifold, we can decompose in each optimization step into the transformation itself $\mathbf{T}(\boldsymbol{\xi})$ and a small increment (could be left or right-multiplied) $\boldsymbol{\delta\xi}$:

$$\mathbf{T}(\boldsymbol{\xi}) := \mathbf{T}(\boldsymbol{\xi})\mathbf{T}(\boldsymbol{\delta\xi})$$

- Gradient descent operates on the auxiliary variable

$$\boldsymbol{\delta\xi} \leftarrow \mathbf{0} - \nabla_{\boldsymbol{\delta\xi}} E(\boldsymbol{\delta\xi})$$

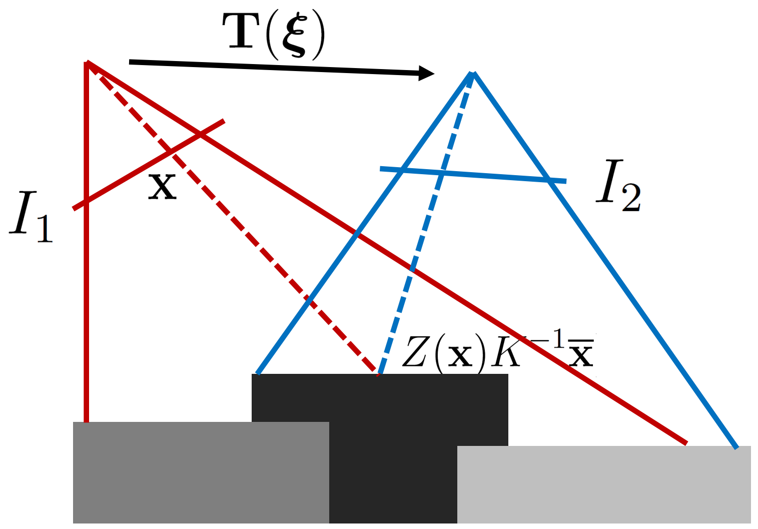
$$\widehat{\boldsymbol{\xi}} \leftarrow \log\left(\exp\left(\widehat{\boldsymbol{\xi}}\right)\exp\left(\widehat{\boldsymbol{\delta\xi}}\right)\right)$$

SE(3) Lie Algebra for Representing Motion

- C++ implementation: Sophus extension library for Eigen, by Hauke Strasdat, https://github.com/strasdat/Sophus

- Further reading on motion representation using the SE(3) Lie algebra:
  - Yi Ma, Stefano Soatto, Jana Kosecka, Shankar S. Sastry. An Invitation to 3-D Vision, Chapter 2: http://vision.ucla.edu/MASKS/
  - http://ingmec.ual.es/~jlblanco/papers/jlblanco2010geometry3D_techrep.pdf
  - http://ethaneade.com/lie.pdf

Dense Direct Image Alignment


$\mathbf{T}(\boldsymbol{\xi})$
I_1
$\mathbf{x}$
I_2
$Z(\mathbf{x})K^{-1}\overline{\mathbf{x}}$

- If we know pixel depth, we can „simulate“ an RGB-D image from a different view point
- Ideally, the warped image is the same like the image taken from that pose:

$$I_1(\mathbf{x}) = I_2(\pi(\mathbf{T}(\boldsymbol{\xi})Z(\mathbf{x})K^{-1}\overline{\mathbf{x}}))$$

- For RGB-D, we have the depth, but want to find the camera motion!

Dense Direct Image Alignment

- Given a camera motion, we can find and compare corresponding pixels through projection.
- We measure in one image a noisy version of the intensity in the other image:

$$I_1(\mathbf{x}) = I_2(\pi(\mathbf{T}(\boldsymbol{\xi}) Z(\mathbf{x}) K^{-1} \overline{\mathbf{x}})) + \epsilon$$

- A simple assumption is Gaussian noise, e.g. if the noise only comes from pixel noise on the chip

$$\epsilon \sim \mathcal{N}(0, \sigma_I^2)$$

- If we further assume that the measurements are stochastically independent at each pixel, we can formulate the joint probability

$$p(\boldsymbol{\xi} \mid I_1, I_2) \propto p(I_1 \mid \boldsymbol{\xi}, I_2) p(\boldsymbol{\xi})$$

$$p(\boldsymbol{\xi} \mid I_1, I_2) \propto \prod_{\mathbf{x} \in \Omega} \mathcal{N}\left(I_1(\mathbf{x}) - I_2(\pi(\mathbf{T}(\boldsymbol{\xi}) Z(\mathbf{x}) K^{-1} \overline{\mathbf{x}})); 0, \sigma_I^2\right)$$

Dense Direct Image Alignment

- Maximum-likelihood estimation problem
- Optimize negative log-likelihood
  - Product becomes a summation
  - Exponentials disappear
  - Normalizers are independent of the pose

$$E(\boldsymbol{\xi}) = \text{const.} + \frac{1}{2} \sum_{\mathbf{x} \in \Omega} \frac{r(\mathbf{x}, \boldsymbol{\xi})^2}{\sigma_I^2}$$

$$r(\mathbf{x}, \boldsymbol{\xi}) = I_1(\mathbf{x}) - I_2(\pi(\mathbf{T}(\boldsymbol{\xi}) Z(\mathbf{x}) K^{-1} \overline{\mathbf{x}}))$$

- This non-linear least squares error function can be efficiently optimized using standard methods (Gauss-Newton, Levenberg-Marquardt)

Least Squares Optimization

- If the residuals would be linear $\boldsymbol{\xi}$ i.e., $r(\boldsymbol{\xi}) = \mathbf{A}\boldsymbol{\xi} + \mathbf{b}$ optimization would be simple, has a closed-form solution
- In this case, the error function and its derivatives are

$$E(\boldsymbol{\xi}) = \frac{1}{2} r(\boldsymbol{\xi})^T \mathbf{W} r(\boldsymbol{\xi})$$

$$\nabla_{\boldsymbol{\xi}} E(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})^T \mathbf{W} r(\boldsymbol{\xi}) = \mathbf{A}^T \mathbf{W} r(\boldsymbol{\xi})$$

$$\nabla^2_{\boldsymbol{\xi}} E(\boldsymbol{\xi}) = \mathbf{A}^T \mathbf{W} \mathbf{A}$$

- Setting the first derivative to zero yields

$$\nabla_{\boldsymbol{\xi}} E(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} E(\boldsymbol{\xi}_0) + \nabla^2_{\boldsymbol{\xi}} E(\boldsymbol{\xi}_0)(\boldsymbol{\xi} - \boldsymbol{\xi}_0) = 0$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}_0 - \nabla^2_{\boldsymbol{\xi}} E(\boldsymbol{\xi}_0)^{-1} \nabla_{\boldsymbol{\xi}} E(\boldsymbol{\xi}_0)$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}_0 - \left(\mathbf{A}^T \mathbf{W} \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{W} r(\boldsymbol{\xi}_0)$$

Non-linear Least Squares Optimization

- In direct image alignment, the residuals are non-linear in $\boldsymbol{\xi}$
- Gauss-Newton method, iterate:
  - Linearize residuals

$$\widetilde{r}(\boldsymbol{\xi}) = r(\boldsymbol{\xi}_0) + \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})(\boldsymbol{\xi} - \boldsymbol{\xi}_0)$$

$$\widetilde{E}(\boldsymbol{\xi}) = \frac{1}{2}\widetilde{r}(\boldsymbol{\xi})^T \mathbf{W} \widetilde{r}(\boldsymbol{\xi})$$

$$\nabla_{\boldsymbol{\xi}} \widetilde{E}(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})^T \mathbf{W} \widetilde{r}(\boldsymbol{\xi})$$

$$\nabla^2_{\boldsymbol{\xi}} \widetilde{E}(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})^T \mathbf{W} \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})$$

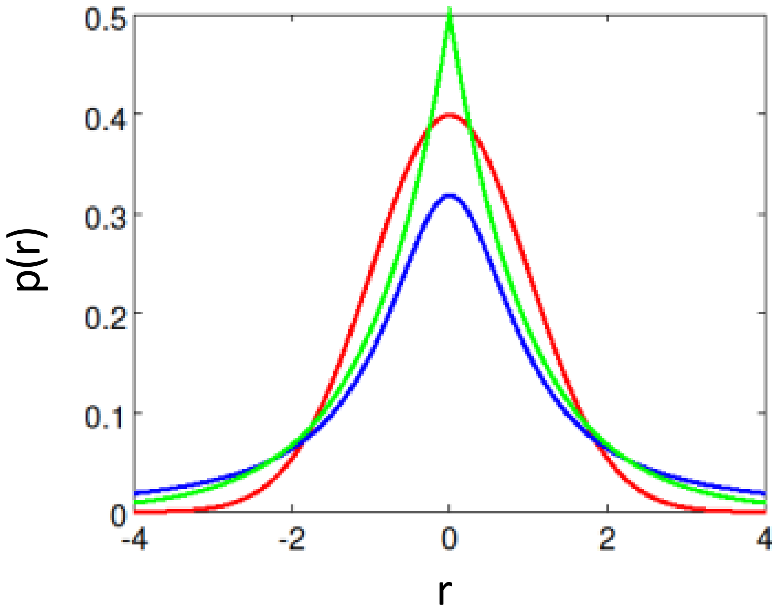
  - Solve linearized system

$$\nabla_{\boldsymbol{\xi}} \widetilde{E}(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} \widetilde{E}(\boldsymbol{\xi}_0) + \nabla^2_{\boldsymbol{\xi}} E(\boldsymbol{\xi}_0)(\boldsymbol{\xi} - \boldsymbol{\xi}_0) = 0$$

$$\boldsymbol{\xi} \leftarrow \boldsymbol{\xi} - \nabla^2_{\boldsymbol{\xi}} \widetilde{E}(\boldsymbol{\xi})^{-1} \nabla_{\boldsymbol{\xi}} \widetilde{E}(\boldsymbol{\xi})$$

$$\boldsymbol{\xi} \leftarrow \boldsymbol{\xi} - \left(\nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})^T \mathbf{W} \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})\right)^{-1} \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})^T \mathbf{W} r(\boldsymbol{\xi})$$

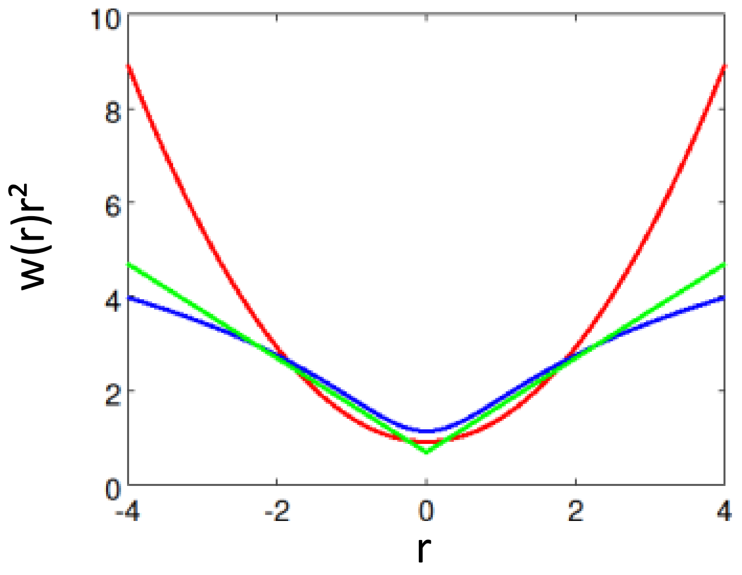
Actual Residual Distribution



- Normal distribution
- Laplace distribution
- Student-t distribution

- The Gaussian noise assumption is not valid
- Many outliers (occlusions, motion, etc.)
- Residuals are distributed with more mass on the larger values

Iteratively Reweighted Least Squares



- Normal distribution
- Laplace distribution
- Student-t distribution

- Can we change the residual distribution in the least squares optimization?
- We can reweight the residuals in each iteration to adapt residual distribution

$$E(\boldsymbol{\xi}) = \frac{1}{2} \sum_{\mathbf{x} \in \Omega} w(r(\mathbf{x}, \boldsymbol{\xi})) \frac{r(\mathbf{x}, \boldsymbol{\xi})^2}{\sigma_I^2}$$

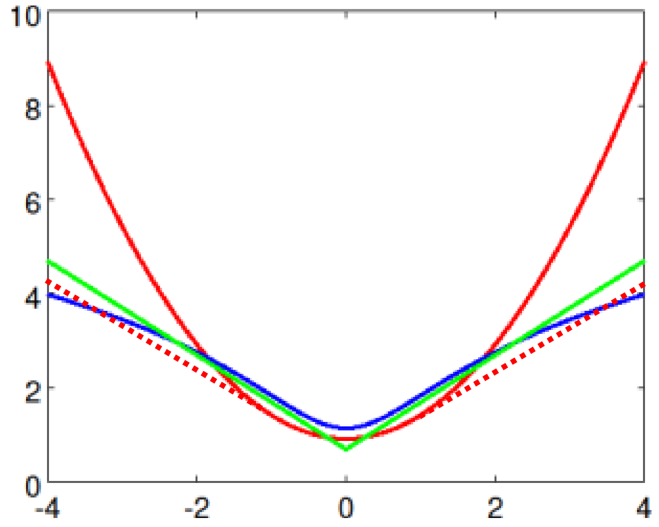
E.g., for Laplace distribution:

$$w(r(\mathbf{x}, \boldsymbol{\xi})) = |r(\mathbf{x}, \boldsymbol{\xi})|^{-1}$$

Huber-Loss

- Huber-loss „switches“ between normal (locally at mean) and Laplace distribution

$$\|r\|_\delta = \begin{cases} \frac{1}{2}\|r\|_2^2 & \text{if } \|r\|_2 \le \delta \\ \delta\left(\|r\|_1 - \frac{1}{2}\delta\right) & \text{otherwise} \end{cases}$$



🟥 Huber-loss for $\delta = 1$

Linearization of Image Alignment Residuals

- In our direct image alignment case, the linearized residuals are

$$\nabla_{\boldsymbol{\xi}} r(\mathbf{x}, \boldsymbol{\xi}) = -\nabla_{\pi} I_2(\pi(\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}))) \cdot \nabla_{\boldsymbol{\xi}} \pi(\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}))$$

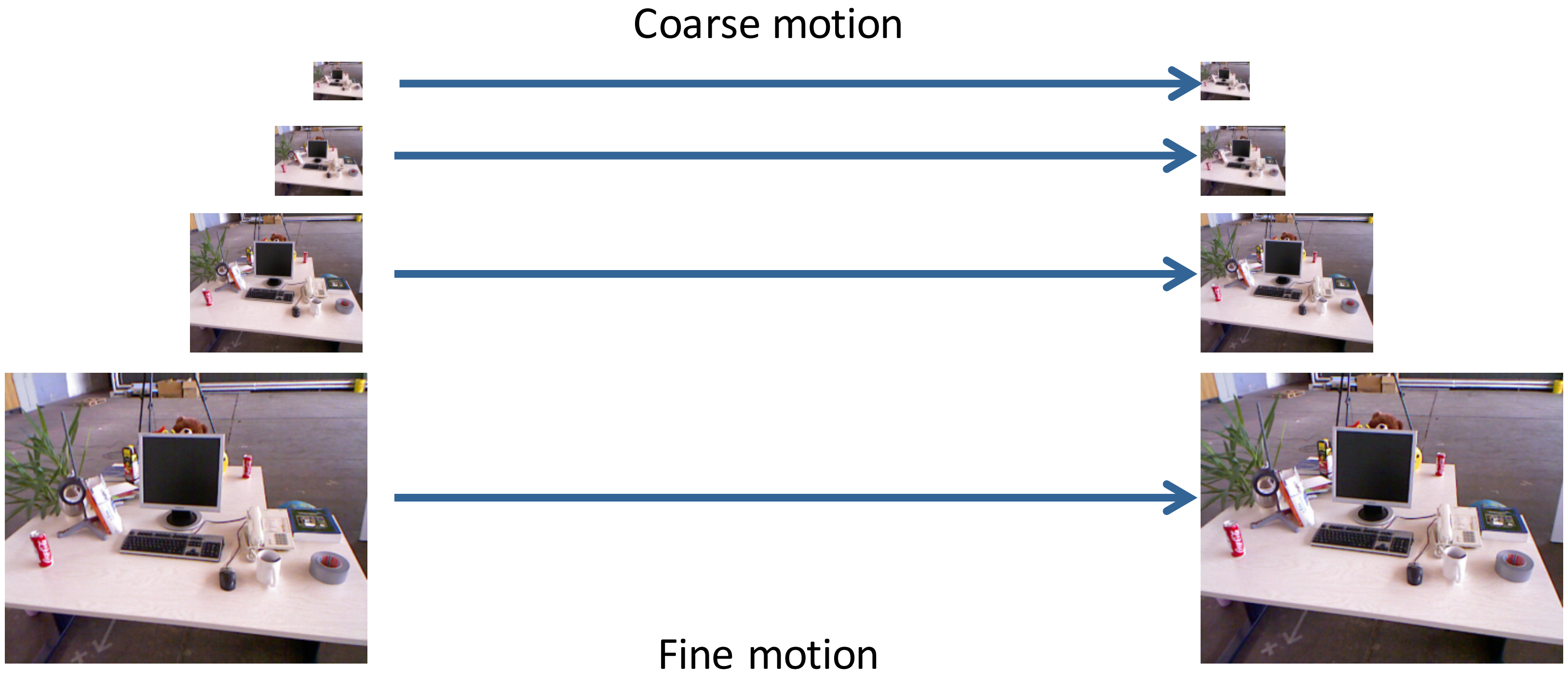
with $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \mathbf{T}(\boldsymbol{\xi}) Z(\mathbf{x}) K^{-1} \overline{\mathbf{x}}$

$$r(\mathbf{x}, \boldsymbol{\xi}) = I_1(\mathbf{x}) - I_2(\pi(\mathbf{p}(\mathbf{x}, \boldsymbol{\xi})))$$

- Linearization is only valid for motions that change the projection in a small image neighborhood (where the gradient hints into the direction)

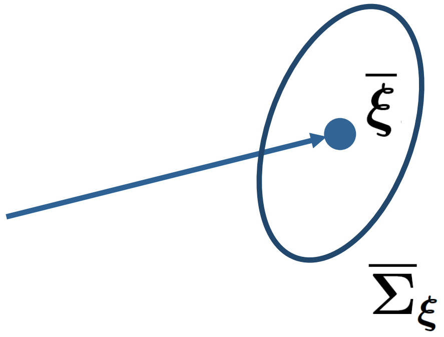
Coarse-To-Fine

- Adapt size of the neighborhood from coarse to fine



Covariance of the Pose Estimate

- Non-linear least squares determines a Gaussian estimate



$$p(\boldsymbol{\xi} \mid I_1, I_2) = \mathcal{N}\left(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\Sigma}}_{\boldsymbol{\xi}}\right)$$

$$\bar{\boldsymbol{\Sigma}}_{\boldsymbol{\xi}} = \left(\nabla_{\boldsymbol{\xi}} r(\bar{\boldsymbol{\xi}})^T \mathbf{W} \nabla_{\boldsymbol{\xi}} r(\bar{\boldsymbol{\xi}})\right)^{-1}$$

- Due to pose decomposition, we have to change the coordinate frame of the covariance using the adjoint in SE(3)

$$p(\boldsymbol{\xi} \mid I_1, I_2) = \mathcal{N}\left(\bar{\boldsymbol{\xi}}, \mathrm{ad}_{\mathbf{T}(\bar{\boldsymbol{\xi}})} \bar{\boldsymbol{\Sigma}}_{\boldsymbol{\delta\xi}} \mathrm{ad}^T_{\mathbf{T}(\bar{\boldsymbol{\xi}})}\right)$$

$$\bar{\boldsymbol{\Sigma}}_{\boldsymbol{\delta\xi}} = \left(\nabla_{\boldsymbol{\delta\xi}} r(\boldsymbol{\delta\xi} = 0, \bar{\boldsymbol{\xi}})^T \mathbf{W} \nabla_{\boldsymbol{\delta\xi}} r(\boldsymbol{\delta\xi} = 0, \bar{\boldsymbol{\xi}})\right)^{-1}$$

$$\mathrm{ad}_{\mathbf{T}} = \begin{pmatrix} \mathbf{R} & [\mathbf{t}]_\times \mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{pmatrix} \in \mathbb{R}^{6\times 6}$$

Levenberg-Marquardt

- Idea: damp Gauss-Newton algorithm

$$\boldsymbol{\xi} \leftarrow \boldsymbol{\xi} - \left(\nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})^T \mathbf{W} \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi}) + \lambda \mathbf{I}\right)^{-1} \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})^T \mathbf{W} r(\boldsymbol{\xi})$$

- More adaptive component-wise damping:

$$\boldsymbol{\xi} \leftarrow \boldsymbol{\xi} - \Big(\nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})^T \mathbf{W} \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi}) + \lambda\, \mathrm{diag}(\nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})^T \mathbf{W} \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi}))\Big)^{-1} \nabla_{\boldsymbol{\xi}} r(\boldsymbol{\xi})^T \mathbf{W} r(\boldsymbol{\xi})$$

- Hybrid between Newton method ($\lambda = 0$) and gradient descent with step size $1/\lambda$ (for $\lambda \rightarrow \infty$)

- Start with e.g. $\lambda = 0.1$ and update λ in each iteration
- decrease λ in case of successful update (decreased error)
- increase λ in case of unsuccessful update (increased error)

Part 3: Lessons Learned

- The SE(3) Lie algebra is an elegant way of motion representation, especially for gradient-based optimization of motion parameters
- Non-linear least squares optimization is a versatile tool that can be applied for direct image alignment
- Iteratively Reweighted Least Squares allows for overcoming the limitation of basic least squares on the Gaussian residual distribution/L2 loss on the residuals
- Dense RGB-D odometry through direct image alignment can be implemented in a non-linear least squares framework.
  - The linear approximation of the residuals requires a coarse-to-fine optimization scheme
  - Non-linear least squares also provides the pose covariance

Questions ?