Chapter 3
Variational Calculus

Computer Vision I: Variational Methods

Winter 2016/17

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Variational Methods

Variational methods are a specific class of optimization methods. The key idea is to define cost functionals over a continuous solution space and to compute optima by solving the corresponding extremality principle.

Variational methods allow to solve respective problems in a mathematically transparent manner. Instead of performing a heuristic sequence of processing steps one starts by defining what properties a solution should have. Once these are fixed, the appropriate algorithm can be derived “automatically”.

Variational methods are particularly suited for infinite-dimensional problems. They are among the top performing methods for:

- image denoising, deblurring, super-resolution
- image segmentation
- motion estimation
- dense 3D reconstruction
- tracking

Advantages of Optimization Methods

Optimization methods have many advantages over traditional multi-step approaches (such as the Canny edge detector):

- A mathematical analysis of the cost function allows statements regarding the existence, uniqueness and stability of solutions to a given problem.
- In traditional multistep processes the interplay of consecutive steps is often complex and intransparent. It is typically unclear how modifying or replacing one component affects the subsequent steps.
- Optimization methods are based on transparent and explicitly formulated assumptions, with no “hidden” assumptions.
- In general, optimization methods have fewer parameters. The meaning of each parameter is fairly obvious.
- Optimization methods are easily combined in a transparent manner (by adding respective cost functions).

A Simple Example: Image Denoising

Let $f : \Omega \to \mathbb{R}$ be an input image corrupted by noise. The goal is to compute a denoised version u of the image f.

The desired function u should fulfill two criteria:

- The function u should be similar to f.
- The function u should be spatially smooth.

Both criteria can be combined in the following cost function (or energy):

$$E(u) = E_{data}(u, f) + \lambda \, E_{smoothness}(u),$$

where the first term measures the similarity of u and f and the second term measures the smoothness of u. A weighting or regularization parameter $\lambda \geq 0$ specifies the relative importance of smoothness versus data fit.

Most variational approaches have the above form. The merely differ in how the similarity term (data term) and the smoothness term (regularizer) are defined.

Discrete Denoising in 1D

Let us assume for now that f is discrete in one spatial variable, i.e. $f = \{f_1, f_2, \ldots, f_n\}$ is simply a sequence of brightness values $f_i \in \mathbb{R}$. We seek an approximation $u = \{u_1, \ldots, u_n\}$.

The data term which measures similarity of f and u can for example be written as:

$$E_{data}(u) = \frac{1}{2} \sum_{i=1}^{n} (f_i - u_i)^2,$$

which means that we measure the overall brightness difference as a sum of squared differences (SSD).

The smoothness term can for example be written as:

$$E_{smooth}(u) = \frac{1}{2} \sum_{i=1}^{n-1} (u_i - u_{i+1})^2,$$

which means that we measure the sum of squared differences for all neighboring brightness values.

Discrete Denoising in 1D

The total energy is thus:

$$E_\lambda(u) = \frac{1}{2}\sum_{i=1}^{n}(f_i - u_i)^2 + \frac{\lambda}{2}\sum_{i=1}^{n-1}(u_i - u_{i+1})^2,$$

Larger values of λ imply that the smoothness of the solution should play a bigger role.

A solution to the above denoising problem is a function $\hat{u}$ which minimizes the above energy:

$$\hat{u} = \arg\min_u E_\lambda(u).$$

Variational methods determine functions which fulfill the extremality principle:

$$\frac{dE_\lambda(u)}{du} = 0, \quad \Leftrightarrow \quad \frac{\partial E_\lambda(u)}{\partial u_i} = 0 \quad \forall i \in [1, n]$$

Discrete Denoising in 1D

The extremality condition for each pixel is therefore:

$$\frac{\partial E_\lambda(u)}{\partial u_1} = (u_1 - f_1) + \lambda(u_1 - u_2) = 0,$$

$$\frac{\partial E_\lambda(u)}{\partial u_i} = (u_i - f_i) + \lambda(2u_i - u_{i-1} - u_{i+1}) = 0, \quad \forall i \in [2, n-1],$$

$$\frac{\partial E_\lambda(u)}{\partial u_n} = (u_n - f_n) + \lambda(u_n - u_{n-1}) = 0.$$

These conditions form a system of linear equations:

$$M_\lambda u = \begin{pmatrix} 1+\lambda & -\lambda & & & \\ -\lambda & 1+2\lambda & -\lambda & & \\ & \ddots & \ddots & \ddots & \\ & & -\lambda & 1+2\lambda & -\lambda \\ & & & -\lambda & 1+\lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}$$

Discrete Denoising in 1D

- The matrix M_λ is tridiagonal.
- The inverse matrix M_λ^{-1} exists. It is dense and has only non-negative entries.
- The above system of equations can be solved in linear time using Gaussian elimination (Thomas' algorithm).
- The minimizer is unique because $E(u)$ is strictly convex.
- Reminder:
  - A set is called convex, if for any pair of points in the set, the connecting line is also contained in the set.
  - A function $E(u)$ is called convex, if its epigraph is convex, i.e. if for any two points on the graph of E the function values are below the connecting line.
  - $E(u)$ is called strictly convex (streng konvex) if the function values are strictly below the connecting line:

$$E((1-\alpha)u_1+\alpha u_2) < (1-\alpha)E(u_1)+\alpha E(u_2) \quad \forall u_1, u_2 \, \forall \alpha \in (0,1).$$

Proof of convexity

The function $h(s) = s^2$ is convex:

$$h\big((1-\alpha)s_1+\alpha s_2\big) < (1-\alpha)h(s_1)+\alpha h(s_2), \quad \forall s_1, s_2, \ \forall\alpha \in (0,1).$$

For any $u = (u_1, \ldots, u_n)$ and $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)$ and for any $\alpha \in (0,1)$ we therefore have:

$$\begin{aligned}
E_\lambda((1-\alpha)u + \alpha\tilde{u}) &= \frac{1}{2}\sum_{i=1}^{n}\Big(f_i - ((1-\alpha)u_i + \alpha\tilde{u}_i)\Big)^2 \\
&\quad + \frac{\lambda}{2}\sum_{i=1}^{n-1}\Big(((1-\alpha)u_i + \alpha\tilde{u}_i) - ((1-\alpha)u_{i+1} + \alpha\tilde{u}_{i+1})\Big)^2 \\
&= \frac{1}{2}\sum_{i=1}^{n}\Big((1-\alpha)(f_i - u_i) + \alpha(f_i - \tilde{u}_i)\Big)^2 \\
&\quad + \frac{\lambda}{2}\sum_{i=1}^{n-1}\Big(((1-\alpha)(u_i - u_{i+1}) + \alpha\big(\tilde{u}_i - \tilde{u}_{i+1}\big)\Big)^2 \\
&< (1-\alpha)E_\lambda(u) + \alpha E_\lambda(\tilde{u}).
\end{aligned}$$

Discrete Denoising (d-dim.)

Index all pixels of the d-dim volume with index $i \in [1, \dots, N]$, where $N = n_1 \cdot n_2 \cdots n_d$.

Variational denoising of an image f:

$$E_\lambda(u) = \frac{1}{2} \sum_{i=1}^{N} (f_i - u_i)^2 + \frac{\lambda}{2} \sum_{i=1}^{N} \sum_{j \in \mathcal{N}(i)} (u_i - u_j)^2,$$

where $\mathcal{N}(i)$ denotes the neighborhood of pixel i.

Again E is strictly convex. The condition for (global) optimality is:

$$\frac{dE_\lambda(u)}{du_i} = (u_i - f_i) + \lambda \sum_{j \in \mathcal{N}(i)} (u_i - u_j) = 0 \qquad \forall i$$

In the higher-dimensional case, this gives rise to a large-scale linear programming problem.

Large Scale Linear Programming

For dimensions ($d = 2, 3, ...$), the linear equation system

$$M_\lambda u = f$$

is quite large.

Most entries of the matrix M_λ are 0 (sparse matrix). Yet, its inverse is typically difficult to compute or even to store in memory.

There exist numerous standard solvers for large linear systems. The best known ones are the Jacobi method and the Gauss-Seidel method.

In the following, we will discuss the Jacobi solver, the Gauss-Seidel solver and various extensions.

The Jacobi Method

The Jacobi method converges if the matrix $M \equiv M_\lambda$ is strictly diagonally dominant, i.e. if for any row of the matrix the absolute value of the diagonal element is larger than the sum of absolute values of the off-diagonal elements:

$$|m_{ii}| > \sum_{j \neq i} |m_{ij}| \quad \forall i$$

Decompose the matrix $M = D + A$ into its diagonal part D and the offdiagonal part A. Then:

$$Mu = (D + A)u = f \quad \Leftrightarrow \quad Du = f - Au$$

Initialize with an aribtrary function u^0 and iterate:

$$u^{(k+1)} = D^{-1}(f - Au^{(k)}), \quad k = 0, 1, 2, 3, \ldots$$

Low computational and memory cost, easily parallelizable.

We can use the residuum $r^{(k)} := Mu^{(k)} - f$ as a termination criterion: $\frac{|r^{(k)}|}{|r^{(0)}|} < \epsilon$.

Application to Image Denoising

- Separate diagonal part D:

$$u_i + \lambda|\mathcal{N}(i)|u_i = f_i + \lambda \sum_{j \in \mathcal{N}(i)} u_j \quad \forall i,$$

  where $|\mathcal{N}(i)|$ is the number of neighbors of pixel i.

- Iteration:

$$u_i^{(k+1)} = \frac{f_i + \lambda \sum\limits_{j \in \mathcal{N}(i)} u_j^{(k)}}{1 + \lambda|\mathcal{N}(i)|}$$

- Appropriate initialization:

$$u^{(0)} = f$$

- Intuitive Interpretation: For each pixel i, the above iteration induces a weighted averaging of brightness values in its neighborhood $\mathcal{N}(i)$. The weights sum to 1.

Alternative Methods

Besides the Jacobi method, there are other methods for solving linear equation systems of the form $Mu = f$. In particular:

- Gauss-Seidel method: Updates pixel values sequentially always using the most recent values:

$$u_i^{(k+1)} = \frac{1}{m_{ii}} \left(f_i - \sum_{j<i} m_{ij} u_j^{(k+1)} - \sum_{j>i} m_{ij} u_j^{(k)} \right)$$

  Faster than the Jacobi method. But: not parallelizable, solution depends on the order of updates.

- SOR method (Successive Over-Relaxation): Extrapolate the Gauss-Seidel updates linearly for faster convergence:

$$u_i^{(k+1)} = \omega \bar{u}_i^{(k+1)} + (1 - \omega) u_i^{(k)}$$

  where $\bar{u}_i^{(k+1)}$ is the current solution of the Gauss-Seidel method, and $\omega \in [1, 2)$ the extrapolation factor ($\omega = 1$: Gauss-Seidel).

Alternative Methods

There exists a multitude of numerical strategies to accelerate algorithms. Two common examples are:

- Preconditioned Conjugate Gradients: Modification of the gradient descent or steepest descent which minimizes along orthogonal directions. Preconditioning can additionally improve the convergence rate.
- Multigrid Methods: Solve the equation system starting from a coarse-grid representation and use solution as initalization on respective finer grids.

Which combination of methods leads to the fastest solution depends on the type of optimization problem / cost function and the available hardware.

For example, the Gauss-Seidel and SOR methods are typically faster than the Jacobi method, but they are not directly parallelizable.

Spatially Continuous Variational Approaches

A spatially continuous variational approach to image denoising and restoration looks as follows:

Given an image $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ find a smooth approximation $u : \Omega \to \mathbb{R}$ of this image. This can be determined by minimizing a cost function of the form:

$$E(u) = \frac{1}{2} \int \big(u(x) - f(x)\big)^2 \, dx \; + \; \frac{\lambda}{2} \int |\nabla u(x)|^2 \, dx$$

This is the spatially continuous analogue of the previously discussed discrete formulation.

Such a mapping which assigns a real number $E(u)$ to a function $u(x)$ is also referred to as a functional.

How does one minimize the above functional with respect to the function u?

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Functional Minimization

A functional is a mapping E which assigns to each element of a vector-space (to each function u) an element from the underlying field (a number).

Let

$$E(u) = \int \mathcal{L}(u, u')\, dx$$

be a functional, where $u' = \frac{du}{dx}$ is the derivative of the function u. (In physics $\mathcal{L}$ is called the Lagrange density).

Example: $\mathcal{L}(u, u') = \frac{1}{2}\big(u(x) - f(x)\big)^2 + \frac{\lambda}{2}|u'(x)|^2$.

Just as with real-valued functions defined on $\mathbb{R}^n$ the necessary condition for extremality of the functional E states that the derivative with respect to u must be 0.

Yet how does one define and compute the derivative of a functional $E(u)$ with respect to the function u?

The Gâteaux Derivative

There are several ways to introduce functional derivatives. The following definition goes back to works of the French mathematician R. Gâteaux († 1914) which were published posthumously in 1919: `http://archive.numdam.org/ARCHIVE/BSMF/BSMF_1919__47_/BSMF_1919__47__47_1/BSMF_1919__47__47_1.pdf`

The Gâteaux derivative extends the concept of directional derivative to infinite-dimensional spaces.

The derivative of the functional $E(u)$ in direction $h(x)$ is defined as:

$$\boxed{\left.\frac{dE(u)}{du}\right|_h = \lim_{\epsilon \to 0} \frac{E(u+\epsilon h) - E(u)}{\epsilon}}$$

As in finite dimensions, this directional derivative can be interpreted as the projection of the functional gradient on the respective direction. We can therefore write:

$$\left.\frac{dE(u)}{du}\right|_h = \left\langle \frac{dE(u)}{du}, h \right\rangle = \int \frac{dE(u)}{du}(x)\, h(x)\, dx$$

The Gâteaux Derivative

For functionals of the canonical form: $E(u) = \int \mathcal{L}(u, u')\, dx$ the Gâteaux derivative is given by

$$\left.\frac{dE(u)}{du}\right|_h = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(E(u + \epsilon h) - E(u)\big)$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int \big(\mathcal{L}(u + \epsilon h, u' + \epsilon h') - \mathcal{L}(u, u')\big)\, dx$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int \left(\left(\mathcal{L}(u, u') + \frac{\partial \mathcal{L}}{\partial u}\epsilon h + \frac{\partial \mathcal{L}}{\partial u'}\epsilon h' + o(\epsilon^2)\right) - \mathcal{L}(u, u')\right) dx$$

$$= \int \left(\frac{\partial \mathcal{L}}{\partial u} h + \frac{\partial \mathcal{L}}{\partial u'} h'\right) dx$$

$$= \int \left(\frac{\partial \mathcal{L}}{\partial u} h - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u'} h\right) dx \qquad \text{(partial int., } h = 0 \text{ on boundary)}$$

$$= \int \underbrace{\left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u'}\right)}_{\frac{dE}{du}} h(x)\, dx.$$

Euler-Lagrange Equation

Thus the derivative of the functional $E(u)$ in direction h is:

$$\frac{dE(u)}{du}\bigg|_h = \int \underbrace{\left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u'}\right)}_{\frac{dE}{du}} h(x)\, dx.$$

As a necessary condition for minimality of the functional $E(u)$ the variation of E in any direction $h(x)$ must vanish. Therefore at the extremum we have:

$$\boxed{\frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u'} = 0}$$

This condition is called the Euler-Lagrange equation.

Example: For $\mathcal{L}(u, u') = \frac{1}{2}\big(u(x) - f(x)\big)^2 + \frac{\lambda}{2}|u'(x)|^2$, we get:

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u'} = \big(u(x) - f(x)\big) - \frac{d}{dx}\big(\lambda u'(x)\big) = u - f - \lambda u'' = 0$$

Functional Minimization

The Euler-Lagrange equation is a differential equation which forms the necessary condition for minimality.

The central idea of variational methods is to compute solutions to the respective Euler-Lagrange equation.

This can be done in several ways. For example, one can discretize the function u on a set of points $\{x_1, \dots, x_n\}$ and subsequently try to solve for the values $u(x_i)$. For quadratic cost functions, the arising set of linear equations can be solved using the discussed iterative algorithms (Jacobi, Gauss-Seidel,...). In general, however, the Euler-Lagrange equation will not be linear.

For general (non-quadratic) energies, one can start with an initial guess $u_0(x)$ of the solution and iteratively improve the solution. Such methods are called descent methods.

How can one iteratively improve a given solution?

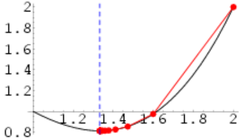
Gradient Descent

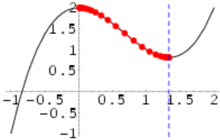
Gradient descent or steepest descent is a particular descent method where in each iteration one chooses the direction in which the energy decreases most. The direction of steepest descent is given by the negative energy gradient.

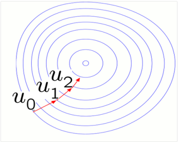
To minimize a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$, the gradient descent for $f(u)$ is defined by the differential equation:

$$\begin{cases} u(0) = u_0 \\ \frac{du}{dt} = -\frac{df}{du}(u) \end{cases}$$

Discretization: $u_{t+1} = u_t - \epsilon \frac{df}{du}(u_t), \qquad t = 0, 1, 2, \ldots.$







Gradient Descent

For minimizing functionals $E(u)$, the gradient descent is done analogously.

For the functional $E(u) = \int \mathcal{L}(u, u')\, dx$, the gradient is given by:

$$\frac{dE}{du} = \frac{d\mathcal{L}}{du} - \frac{d}{dx}\frac{d\mathcal{L}}{du'}.$$

Therefore the gradient descent is given by:

$$\begin{cases} u(x,0) = u_0(x) \\ \dfrac{\partial u(x,t)}{\partial t} = -\dfrac{dE}{du} = -\dfrac{d\mathcal{L}}{du} + \dfrac{\partial}{\partial x}\dfrac{d\mathcal{L}}{du'}. \end{cases}$$

For $\mathcal{L}(u, u') = \frac{1}{2}\left(u - f\right)^2 + \frac{\lambda}{2}|u'|^2$, this means:

$$\frac{\partial u}{\partial t} = (f - u) + \lambda u'' = (f - u) + \lambda \Delta u.$$

If the gradient descent converges, i.e. $\partial_t u = -\frac{dE}{du} = 0$, then we have found a solution to the Euler-Lagrange equation.

Diffusion with Data Term

$$E(u) = \int (f - u)^2 dx + \lambda \int |\nabla u|^2 \, dx \rightarrow \min.$$

$$E(u) = \lambda \int |\nabla u|^2 \, dx \rightarrow \min.$$

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Addendum: Boundary Conditions

When deriving the Euler-Lagrange equations we only considered perturbations $h(x)$ which are 0 on the boundary.

Without this assumption, Gâteaux's directional derivative is:

$$\left.\frac{dE(u)}{du}\right|_h = \cdots = \int_a^b \left(\frac{\partial \mathcal{L}}{\partial u} h + \frac{\partial \mathcal{L}}{\partial u'} h' \right) dx$$

$$= \int_a^b \left(\frac{\partial \mathcal{L}}{\partial u} h - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} h \right) dx + \left(\frac{\partial \mathcal{L}}{\partial u'} h(x) \right)_a^b$$

$$= \int_a^b \left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} \right) \, h(x) \, dx + \left(\frac{\partial \mathcal{L}}{\partial u'} h(x) \right)_a^b = 0$$

$$\Rightarrow \begin{cases} 1) & \dfrac{dE}{du} = \dfrac{\partial \mathcal{L}}{\partial u} - \dfrac{d}{dx} \dfrac{\partial \mathcal{L}}{\partial u'} = 0 \\[2ex] 2) & \left(\dfrac{\partial \mathcal{L}}{\partial u'} h(x) \right)_a^b = 0 \end{cases}$$

Addendum: Boundary Conditions

Depending on the application one can distinguish two kinds of boundary conditions:

- Dirichlet boundary conditions: The function $u(x)$ is fixed on the boundary ($u_r(x)$), i.e. $h(x) = 0$ on the boundary. One only considers variations of $u(x)$ inside the domain:

$$\begin{cases} \dfrac{dE}{du} = \dfrac{\partial \mathcal{L}}{\partial u} - \dfrac{d}{dx}\dfrac{\partial \mathcal{L}}{\partial u'} = 0 \\ u(x)\Big|_{\text{boundary}} = u_r(x) \end{cases}$$

- Von Neumann boundary conditions: One additionally allows for variations of $u(x)$ on the boundary:

$$\begin{cases} \dfrac{dE}{du} = \dfrac{\partial \mathcal{L}}{\partial u} - \dfrac{d}{dx}\dfrac{\partial \mathcal{L}}{\partial u'} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial u'}\Big|_{\text{boundary}} = 0 \end{cases}$$

Diffusion as Gradient Descent

Many diffusion equations (albeit not all) can be derived as the gradient descent on a specific energy.

The energy:

$$E(u) = \int \mathcal{L}(u, \nabla u)\, dx = \frac{1}{2} \int g(x) |\nabla u(x)|^2 \, dx$$

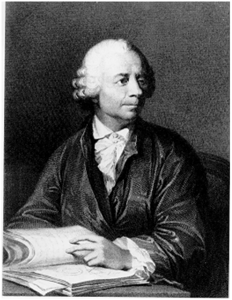
leads to the gradient descent:

$$\frac{\partial u(x,t)}{\partial t} = -\frac{dE}{du} = -\frac{\partial \mathcal{L}}{\partial u} + \nabla \frac{\partial \mathcal{L}}{\partial \nabla u} = \nabla \Big(g(x)\, \nabla u \Big)$$

This equation corresponds to an inhomogeneous diffusion with diffusivity $g(x)$.

In other words, the above inhomogeneous diffusion process is nothing but a steepest descent on the weighted smoothness energy.

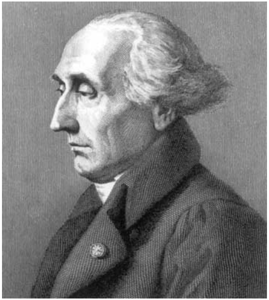
Leonhard Euler



Leonhard Euler (1707 – 1783)

- Published 886 papers and books, most of them in the last 20 years of his life. Considered the greatest mathematician of the 18th century.
- Major contributions: Euler number *e*, Euler angles, Euler formula, Euler theorem, Euler equations (for fluid flows), Euler-Lagrange equations,...
- 13 children

Joseph-Louis Lagrange



Joseph-Louis Lagrange (1736 – 1813)

- born Giuseppe Lodovico Lagrangia (in Turin). self-taught.
- With 17 years: Professor for mathematics in Turin.
- Later in Berlin (1766-1787) and Paris (1787-1813).
- 1788: *La Méchanique Analytique*.
- 1800: *Leçons sur le calcul des fonctions*.