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Chapter 8
Level Set Methods

Computer Vision I: Variational Methods

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Explicit versus Implicit Shape Representations

Shape optimization plays an important role not only in image segmentation, but also in computational physics, fluid mechanics, optimal design and computer graphics.

Gradient descent on respective functionals $E(C)$ leads to an evolution of the boundary in normal direction, which can be implemented explicitly or implicitly. Explicit boundary evolution has the following strengths (+) and weaknesses (-):

+ Explicit evolutions are runtime and memory efficient, allowing a fast evolution of highly detailed boundaries.
+ Prior shape knowledge can be imposed directly on the evolving boundary.
- The numerical propagation of explicit boundaries is prone to instabilities, as self-intersections have to be avoided and regridding of control points may be necessary.
- Respective functionals are typically not convex with respect to the boundary C. Hence solutions are typically only locally optimal.

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Explicit Curve Evolution

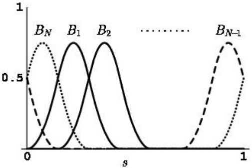
Gradient descent on $E(C)$ leads to an evolution of the curve C

$$\dot{C} = \frac{dC}{dt} = Fn,$$

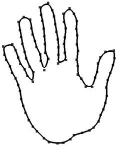
with some speed F in direction of the outer normal n. A parametric representation of the curve as a spline is given by:

$$C(s,t) = \sum_{j=1}^{n} x_j(t)\, B_j(s),$$

with control points $x_1, \dots, x_n \in \mathbb{R}^2$ and basis functions $B_1, \dots, B_n$:



spline basis functions



spline & control points

Explicit Curve Evolution

Inserting the spline representation into the evolution equation gives:

$$\dot{C} = \sum_j \dot{x}_j(t)\, B_j(s) = F\, n$$

Projection onto the basis function B_k leads to:

$$\langle B_k, \dot{C} \rangle = \sum_j \dot{x}_j(t)\, \langle B_k, B_j \rangle = \langle B_k, F\, n \rangle = \int B_k(s) F(s) n(s) ds.$$

This is a linear equation system in $\dot{x}_j$, namely:

$$\boldsymbol{B} \dot{\boldsymbol{x}} = \boldsymbol{q}, \quad \text{with } \boldsymbol{B}_{kj} = \langle B_k, B_j \rangle, \text{ and } \boldsymbol{q}_k = \langle B_k, F\, n \rangle.$$

The temporal evolution of control points is given by:

$$\dot{\boldsymbol{x}} = \boldsymbol{B}^{-1}\, \boldsymbol{q}.$$

Time discretization leads to an update of the control points $\boldsymbol{x}$:

$$\boldsymbol{x}(t + \tau) = \boldsymbol{x}(t) + \tau\, \dot{\boldsymbol{x}}(t) = \boldsymbol{x}(t) + \tau\, \boldsymbol{B}^{-1}\, \boldsymbol{q}(t).$$

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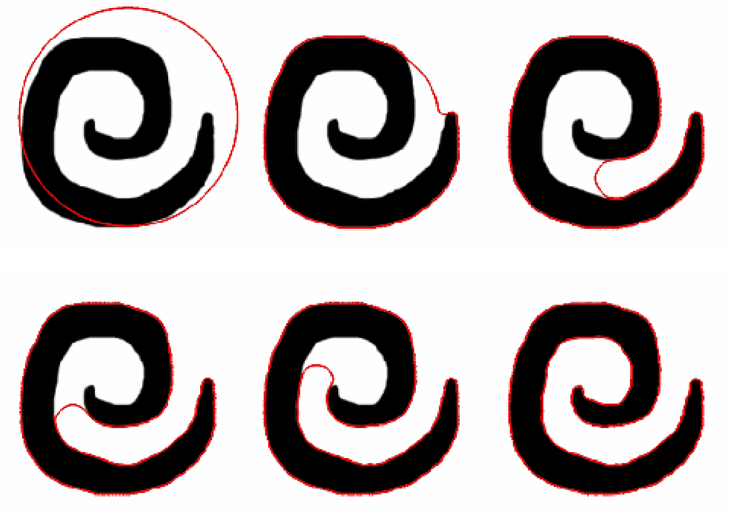

Cremers et al., “Diffusion Snakes”, IJCV 2002

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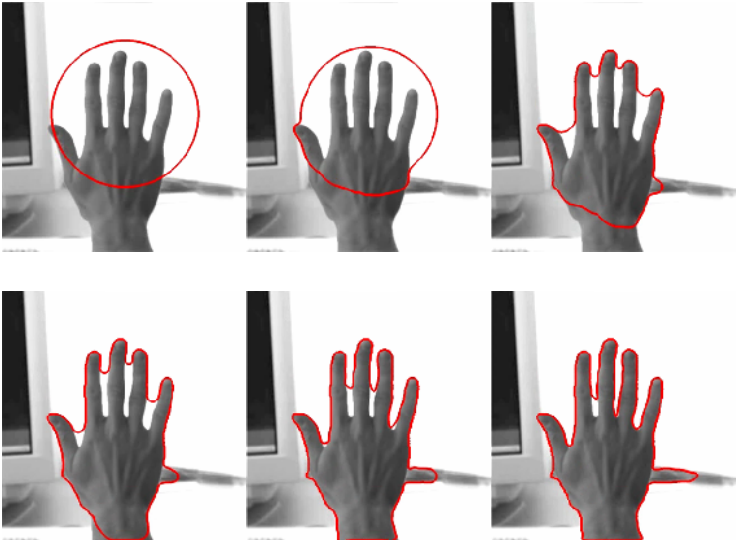

Cremers et al., “Diffusion Snakes”, IJCV 2002

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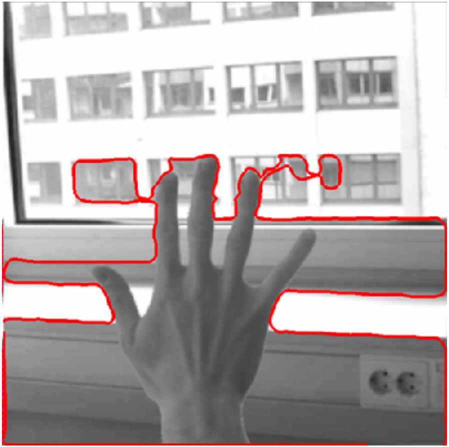
Fixed Curve Topology



By construction, parametric curves have a fixed topology (typically a single closed curve). Without additional splitting or merging heuristics, the curve topology will not change during the curve evolution.

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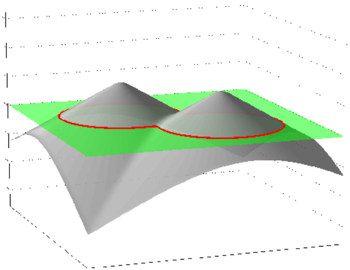
Implicit Shape Representation

Alternatively to an explicit boundary representation, one can represent boundaries C implicitly, for example as the zero level set of an embedding function $\phi : \Omega \to \mathbb{R}$:

$$C = \{x \in \Omega \mid \phi(x) = 0\}.$$



This has several advantages:

- The representation does not require a choice of parameterization.
- The topology of the curve is not fixed.

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Curve Topology is not Constrained

The implicitly represented boundary (red curve) can split and merge freely.

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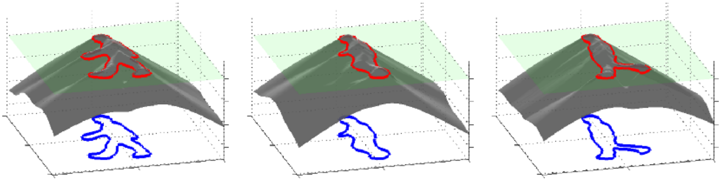
The Level Set Method

The evolution of curves by means of a *dynamical* embedding function is known as the *level set method*. It was first published by Dervieux and Thomasset 1979 and 1981 (around 183 citations in Dec '16) and was later reinvented by Osher and Sethian 1988 (around 12600 citations in Dec '16).

The key idea is to model the temporal evolution of a curve $C(t)$ using a family of embedding functions $\phi(x, t)$ such that:

$$C(t) = \{x \in \Omega \mid \phi(x, t) = 0\}.$$

The central question is how to evolve the embedding function ϕ such that the implicitly represented boundary C follows a prescribed motion.



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Let the motion of the curve C be given by

$$\frac{dC}{dt} = Fn, \tag{1}$$

with some local speed F along the outer normal n. By definition, for any time the embedding function ϕ is zero at all points of the curve C:

$$\phi(C(t), t) = 0 \quad \forall t.$$

As a consequence, the temporal derivative of this expression must be zero:

$$0 = \frac{d}{dt}\phi(C(t), t) = \nabla\phi \cdot \frac{dC}{dt} + \frac{\partial\phi}{\partial t}.$$

We can solve for the temporal evolution of ϕ and insert equation (1) and the definition of the outer normal $n = -\frac{\nabla\phi}{|\nabla\phi|}$:

$$\frac{\partial\phi}{\partial t} = -\nabla\phi \cdot \frac{dC}{dt} = -\nabla\phi \cdot F\,\frac{\nabla\phi}{|\nabla\phi|} = F\,|\nabla\phi|.$$

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The above derivation shows that for a curve evolution with speed F in normal direction, the embedding function at the zero level must follow the equation

$$\frac{\partial \phi}{\partial t} = F \, |\nabla \phi|.$$

Curves (and surfaces) can thus be evolved simply by iterating this partial differential equation called the level set equation. For visualization of the curve or surface, one simply reads out the zero level of $\phi(x, t)$ at any time t. Over time, this curve may undergo splitting and merging.

While the level set equation specifies the motion of ϕ at the boundary, the evolution outside the boundary location can in principle be arbitrary. Typically one imposes that the level set function remains a signed distance function, i.e.:

$$\phi(x, t) = \pm \text{dist}(x, C).$$

where ϕ is positive inside and negative outside the curve.

Level Set Methods for Image Segmentation

The first level set formulations for image segmentation were introduced in the 1990s by Caselles et al. '93, Malladi et al. '95, Caselles et al. '95, Kichenassamy et al. '95, Whitaker '95.

Starting from a variational principle (like the snakes or the Mumford-Shah model) there are two alternative approaches:

- One can derive the gradient descent equation for the curve C (providing the speed function F) and implement it using the above level set equation. This was done to derive a level set method for snake-like energies known as the geodesic active contours (Caselles et al. '95, Kichenassamy et al. '95).
- One can rewrite the variational principle with respect to the level set function ϕ (rather than the curve C) and compute a gradient descent with respect to ϕ. This was proposed by Chan and Vese (2000) to derive a level set method for the Mumford-Shah model.

In the following, we will discuss both of these approaches.

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The Geodesic Active Contours

Consider the edge-based segmentation energy

$$E(C) = \int g(C)\, dC,$$

representing the geodesic length with some edge indicator function g:

$$g(x) = \frac{1}{1 + |\nabla I_\sigma(x)|^2},$$

assiging small values to strong gradients of the smoothed image I_σ.

The gradient descent equation for C is given by:

$$\frac{dC}{dt} = g\kappa n + (n \cdot \nabla g)n,$$

with curvature κ and normal n. The level set equation is:

$$\frac{\partial \phi}{\partial t} = |\nabla \phi| \operatorname{div}\left(g(x)\frac{\nabla \phi}{|\nabla \phi|}\right) = g\,|\nabla \phi| \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) + \nabla g \nabla \phi.$$

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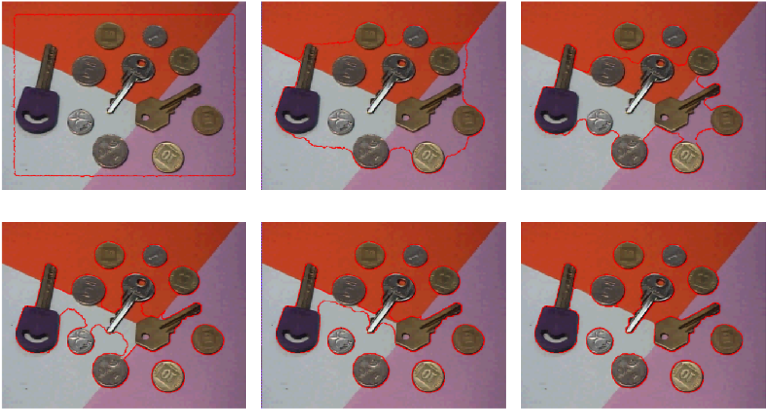
The Geodesic Active Contours



Goldenberg et al., IEEE Trans. on Image Processing 2001

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Goldenberg et al., IEEE Trans. on Image Processing 2001

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The Chan-Vese Level Set Method

In 2000, Chan and Vese proposed a level set method for the Mumford-Shah energy. For the piecewise constant Mumford-Shah model with two regions Ω_1 and $\Omega_2 = \Omega - \Omega_1$, one makes use of the Heaviside step function

$$H\phi \equiv H(\phi) = \begin{cases} 1, & \text{if } \phi > 0 \quad (\text{i.e. } x \in \Omega_1) \\ 0, & \text{else} \quad (\text{i.e. } x \in \Omega_2) \end{cases}$$

With this, we can write the two region model as follows:

$$\begin{aligned} E(\mu_i, \Omega_i) &= \int_{\Omega_1} (I(x) - \mu_1)^2 \, dx + \int_{\Omega_2} (I(x) - \mu_2)^2 \, dx + \nu \, |\partial \Omega_1| \\ &= \int_{\Omega} (I - \mu_1)^2 H\phi + (I - \mu_2)^2 (1 - H\phi) \, dx + \nu \int_{\Omega} |\nabla H\phi| dx \\ &= \int_{\Omega} \Big((I - \mu_1)^2 - (I - \mu_2)^2 \Big) H\phi + (I - \mu_2)^2 \, dx + \nu \int_{\Omega} |\nabla H\phi| dx \end{aligned}$$

The Chan-Vese Level Set Method

Local minimization of the Chan Vese energy can be done by gradient descent. To this end, one assumes that the Heaviside step function is slightly smoothed (to make it differentiable).

Its derivative is then a smoothed delta function:

$$\frac{d}{d\phi} H(\phi) = \delta(\phi).$$

The gradient descent equation can be computed with the standard Euler-Lagrange calculus:

$$\frac{\partial \phi}{\partial t} = -\frac{\partial E}{\partial \phi} = \delta(\phi) \left(\nu \operatorname{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) + (I - \mu_2)^2 - (I - \mu_1)^2 \right)$$

For the smoothed delta function one has various choices, for example:

$$\delta_\epsilon(\phi) = \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + \phi^2}, \qquad \text{with } \epsilon > 0.$$

Additionally, one can perform redistancing to assure that ϕ remains a signed distance function.

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Author: D. Cremers

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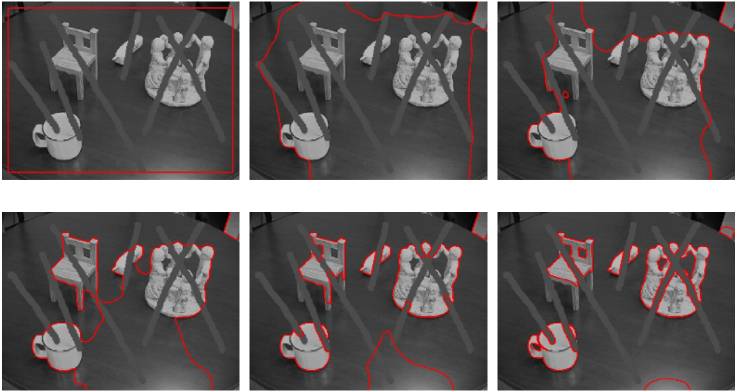

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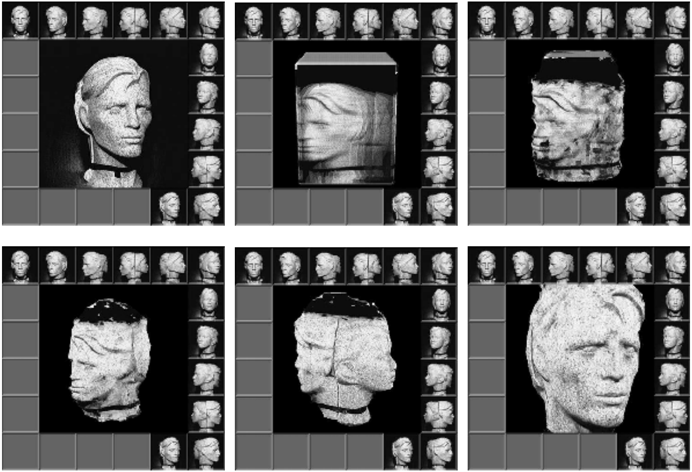

Faugeras, Keriven, IEEE T. on Image Proc. 1998

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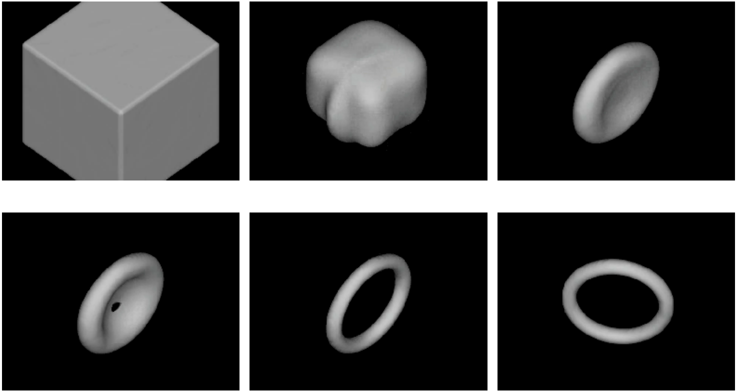

Kolev, Brox, Cremers, DAGM 2006

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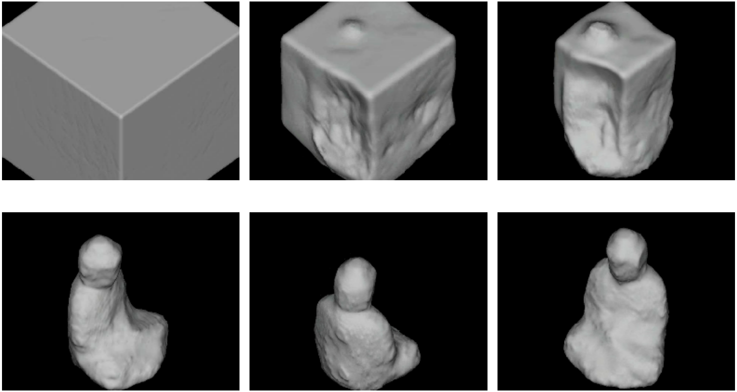

Kolev, Brox, Cremers, DAGM 2006

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Kolev, Brox, Cremers, DAGM 2006

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3 out of 31 input images

Nested level set evolution of surface and segmentation

Jin, Cremers, Wang, Prados, Yezzi, Soatto, IJCV 2007

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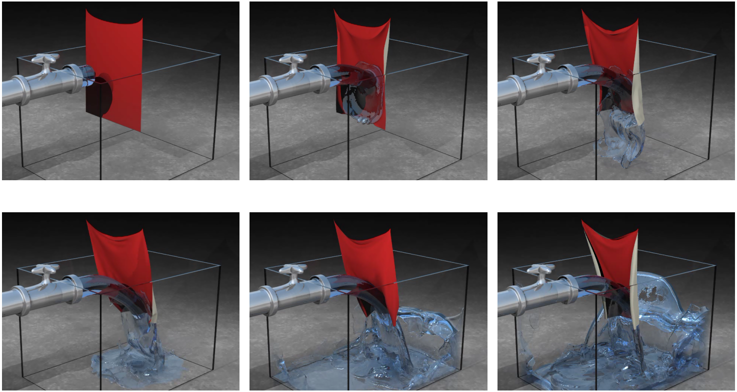

Enright et al., 2003

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