Variational Methods for Computer Vision: Exercise Sheet 1

Exercise: October 24, 2016

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. **Refresher: Multivariate analysis.**

   (a) For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, the *gradient* is defined as $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)^\top$. Calculate the gradients of the following functions.

   i. $f\colon \mathbb{R}^2 \to \mathbb{R}, f(x) = \frac{1}{2}\|x\|_2^2$,

   ii. $f\colon \mathbb{R}^2 \to \mathbb{R}, f(x) = \|x\|_2$.

   Are there any points where the gradient is undefined?

   (b) For a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$, the *Jacobian matrix* at the point $a \in \mathbb{R}^n$ is defined as

$$J_f(a) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix} \in \mathbb{R}^{m\times n}.$$

   Calculate the Jacobian matrix of the following functions:

   i. $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}^2, f(r,\varphi) = (r\cos(\varphi), r\sin(\varphi))^\top$,

   ii. $f\colon \mathbb{R} \to \mathbb{R}^2, f(t) = (r\cos(t), r\sin(t))^\top$.

   (c) For a function $f\colon \mathbb{R}^n \to \mathbb{R}^n$, the *divergence* is defined as $\operatorname{div} f = \sum_{i=1}^n \partial f_i/\partial x_i$. Calculate the divergence of the following functions:

   i. $f\colon \mathbb{R}^2 \to \mathbb{R}^2, f(x,y) = (-y, x)^\top$,

   ii. $f\colon \mathbb{R}^2 \to \mathbb{R}^2, f(x,y) = (x, y)^\top$.

   (d) For a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, the *curl* is defined as $\operatorname{curl} f = \partial f_2/\partial x - \partial f_1/\partial y$. Calculate the curl of function 1(c)i. Prove that the identity $\operatorname{curl}(\nabla f) = 0$ is true for arbitrary $f\colon R^2 \to \mathbb{R}$. Verify the identity with your result from 1(a)i.

   (e) When integrating a function $f\colon S \to \mathbb{R}^n$ over an open subset $S \subset \mathbb{R}^n$ using a parametrization $P \subset \mathbb{R}^n$, $\phi\colon P \to S$, the Jacobian of ϕ has to be taken into account as follows:

$$\int_S f(s)\,\mathrm{d}s = \int_P f(\phi(p))|\det J_\phi(p)|\,\mathrm{d}p.$$

   For $\gamma : [a,b] \to \mathbb{R}^n$, the line integral over a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\int_\gamma f\,\mathrm{d}s = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

   i. Calculate the area enclosed by a circle of radius R.

   ii. Calculate the circumference of a circle of radius R.

   The results from task 1b might be helpful.

(f) The divergence theorem (a special case of Stokes' theorem) states that an integral of the divergence of a function $f\colon S \to \mathbb{R}^n$ over a subset $S \subset \mathbb{R}^n$ can be replaced by an integral over the boundary ∂S of S:

$$\int_S \operatorname{div} f \,\mathrm{d}s = \int_{\partial S} \langle f, n\rangle \,\mathrm{d}s,$$

where $\langle \cdot, \cdot \rangle$ is the dot product and n the unit vector pointing in the direction normal to the boundary.

Convince yourself that this formula holds using f from task 1(c)ii and with S being a disk of radius R.

2. **Convolutions and the Fourier transform.**

(a) Let $f, g, h \in L^1(\mathbb{R})$ be absolutely integrable functions. Consider the convolution of the functions f and g:

$$(f * g)(x) = \int_{\mathbb{R}} f(y)\, g(x-y) \,\mathrm{d}y.$$

Show the following three algebraic identities:

i. $(f * g) * h = f * (g * h)$

ii. $f * g = g * f$

iii. $f * (g + h) = f * g + f * h$

(b) Let $\mathcal{F}$ denote the Fourier transform operator:

$$\mathcal{F}\{f\} := \hat{f}(\nu) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \nu} \,\mathrm{d}x.$$

Prove that the Fourier transform of the convolution of two functions is the same as the pointwise multiplication of the respective Fourier transforms:

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}.$$

What implications does this have for computing the convolution?

(c) Additionally let f and g be continuously differentiable. Show that:

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g = \frac{dg}{dx} * f.$$

The results from 2a might be useful.

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. Start MATLAB and visualize the vector fields from exercise 1(c)i and 1(c)ii. The commands `help meshgrid` and `help quiver` can be useful for that. Explain the intuition behind divergence-free and curl-free vector fields!

2. Download the archive `vmcv_ex01.zip` and unzip it on your home folder. In there should be a file named `coins.png`. Load the unzipped image using the following command:

   ```
   f=double(imread('coins.png'));
   ```

   Show the image using MATLAB's command:

   ```
   figure; imshow(uint8(f));
   ```

3. Compute the convolution of the image with a Gaussian kernel. In theory, the Gaussian distribution is nonzero everywhere, however in practice we restrict ourself to truncated kernels. Set the radius of the kernel to $r = \text{ceil}(3 \times \sigma)$. The discrete convolution is given as:

   $$g(i,j) = (w * f)(i,j) := \sum_{m=-r}^{r} \sum_{n=-r}^{r} w(m,n)\, f(i-m, j-n).$$

   The discrete truncated Gaussian kernel can be written as follows:

   $$w(m,n) \propto \exp\left(-\frac{m^2+n^2}{2\sigma^2}\right)$$

   In order to stay consistent with the continuous formulation of the Gaussian distribution make sure to normalize the kernel function such that the following holds:

   $$\sum_{m=-r}^{r} \sum_{n=-r}^{r} w(m,n) = 1.$$

   For simplicity you can ignore pixels where the mask goes beyond the edge of the image.

4. Let W and H denote respectively the width and height of the input image f. Compute the the gradient $\nabla f = (\partial_x^+ f, \partial_y^+ f)^\mathsf{T}$ of the image using the discretization scheme of forward differences:

   $$(\partial_x^+ f)_{i,j} = \begin{cases} f_{i+1,j} - f_{i,j} & \text{if } i < W \\ 0 & i = W. \end{cases}$$

   $$(\partial_y^+ f)_{i,j} = \begin{cases} f_{i,j+1} - f_{i,j} & \text{if } j < H \\ 0 & j = H. \end{cases}$$

   Notice that the boundary values of the gradient are set to zero.

5. Try solving exercise 4 by avoiding using any for loops this time. Can you tell the difference?

6. Let f_σ be the input image convolved with a Gaussian kernel of standard deviation σ. Compute the magnitude of the Gradient $|\nabla f_\sigma|$ for different values for σ. What do you observe?