Variational Methods for Computer Vision: Exercise Sheet 4

Exercise: November 14, 2014

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. Let $u \in C^2(\Omega; \mathbb{R})$ be a real valued function and $\Omega \subset \mathbb{R}$. And let

   $$E(u) = \int_\Omega \mathcal{L}(u(x), u', u'')\,\mathrm{dx}$$

   be a real valued Gâteaux differentiable functional which depends on:

   $$u(x), \qquad u' = \frac{\mathrm{d}}{\mathrm{dx}}u(x), \qquad \text{and} \qquad u'' = \frac{\mathrm{d}^2}{\mathrm{dx}^2}u(x).$$

   Calculate the Gâteaux derivative of $E(u)$:

   $$\left.\frac{\mathrm{d}E(u)}{\mathrm{du}}\right|_h$$

   for any differentiable direction h.

2. Prove the following identities for $\Omega \subset \mathbb{R}^n$, $f \in C^1(\mathbb{R}^n; \mathbb{R})$, $g \in C^1(\mathbb{R}^n; \mathbb{R}^n)$:

   (a) $\operatorname{div}(fg) = \nabla f \cdot g + f \operatorname{div} g$,

   (b) $\int_\Omega \nabla f \cdot g\,\mathrm{d}x = \int_{\partial\Omega} f \langle g, n\rangle\,\mathrm{d}s - \int_\Omega f \operatorname{div} g\,\mathrm{d}x$,

   where n is the unit vector normal to the boundary $\partial\Omega$. Hint for 2b: divergence theorem on exercise sheet 1.

3. Let $u \in C^1(\mathbb{R}^3; \mathbb{R})$ be a real valued function. And let

   $$E(u) = \int_\Omega \mathcal{L}(u(x), \nabla u(x))\,\mathrm{dx}$$

   be a real valued Gâteaux differentiable functional. Calculate the Gâteaux derivative of $E(u)$. Hint: Use multidimensional integration by parts as derived in question 2b.

4. Let $u \in C^1(\Omega; \mathbb{R})$ be a real valued function with $\Omega \subset \mathbb{R}^2$. Derive the Euler-Lagrange Equation of the following energies:

   (a) The *total variation* of the function u

   $$E_1(u) = \int_\Omega |\nabla u(x)|\,\mathrm{dx}.$$

   (b) The anisotropic total variation

   $$E_2(u) = \int_\Omega \sqrt{\nabla u(x)^T D(x) \nabla u(x)}\,\mathrm{dx},$$

   where $D(x) \in \mathbb{R}^{2\times 2}$ is a real valued matrix.

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. In the lecture you encountered the continuous counterpart of the denoising energy from the previous exercise sheet

$$\frac{\lambda}{2}\int_{\Omega}(u-f)^2 \,\mathrm{dx} + \frac{1}{2}\int_{\Omega}|\nabla u(x)|^2 \,\mathrm{dx},$$

   where $\Omega \subset \mathbb{R}^2$ represents the image domain, $u : \Omega \to \mathbb{R}$ denotes the optimization variable and $f : \Omega \to \mathbb{R}$ stands for the input image. The corresponding Euler-Lagrange equation is given as

$$\lambda(u-f) - \Delta u = 0. \tag{1}$$

   In the finite-dimensional setting, consider the images as vectors $u \in \mathbb{R}^N$, $f \in \mathbb{R}^N$.

   (a) Discretize the continuous gradient operator as $\nabla = (\partial_x^+ \;\; \partial_y^+)^\mathsf{T} \in \mathbb{R}^{2N \times N}$ using forward differences with Neumann boundary conditions, representing it as a sparse matrix (`help spdiags`, `help speye`).

   (b) Solve the linear system arising from (1) using the backslash command (`help \`). Note that the divergence is the negative adjoint of the gradient, thus $\Delta = \operatorname{div} \nabla = -\nabla^\mathsf{T}\nabla$.

2. The Rudin-Osher-Fatemi (ROF) functional[1] was one of the first models for image denoising based on energy minimization. The ROF model posesses the nice property of removing noise in the image while preserving discontinuities. It can be formulated as follows:

$$E_{\text{ROF}}(u) := \frac{\lambda}{2}\int_{\Omega}(u-f)^2 \,\mathrm{dx} + \int_{\Omega}|\nabla u(x)| \,\mathrm{dx},$$

   where again $\Omega \subset \mathbb{R}^2$ represents the image domain, $u : \Omega \to \mathbb{R}$ denotes the optimization variable and $f : \Omega \to \mathbb{R}$ stands for the input image. Hence the ROF model is a weighted sum of the total variation energy and a data similarity measure also called data term.

   (a) In the theoretical exercises we calculated the Euler-Lagrange equation of the total variation of a function u which is a part of the ROF model. Write down the complete optimality condition for the ROF model.

   (b) Obtain an optimal $u^* = \arg\min\; E_{\text{ROF}}(u)$ by applying the gradient descent scheme shown in the lecture, and take care of the nondifferentiability of $|\cdot|$ at 0. Explain the edge preserving properties by looking at the gradient descent equation of E_{ROF}.

[1] L. Rudin, S. Osher, E. Fatemi, Nonlinear total variation based noise removal algorithms, Physica D., 1992.