Variational Methods for Computer Vision: Exercise Sheet 4

Exercise: November 14, 2014

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. Let $u \in C^2(\Omega; \mathbb{R})$ be a real valued function and $\Omega \subset \mathbb{R}$. And let

$$E(u) = \int_\Omega \mathcal{L}(u(x), u', u'')\,\mathrm{dx}$$

be a real valued Gâteaux differentiable functional which depends on:

$$u(x), \qquad u' = \frac{\mathrm{d}}{\mathrm{dx}}u(x), \qquad \text{and} \qquad u'' = \frac{\mathrm{d}^2}{\mathrm{dx}^2}u(x).$$

Calculate the Gâteaux derivative of $E(u)$:

$$\left.\frac{\mathrm{d}E(u)}{\mathrm{du}}\right|_h$$

for any differentiable direction h.

2. Prove the following identities for $\Omega \subset \mathbb{R}^n$, $f \in C^1(\mathbb{R}^n; \mathbb{R})$, $g \in C^1(\mathbb{R}^n; \mathbb{R}^n)$:

   (a) $\mathrm{div}(fg) = \nabla f \cdot g + f\,\mathrm{div}\,g$,

   (b) $\int_\Omega \nabla f \cdot g\,\mathrm{d}x = \int_{\partial\Omega} f\langle g, n\rangle\,\mathrm{d}s - \int_\Omega f\,\mathrm{div}\,g\,\mathrm{d}x$,

   where n is the unit vector normal to the boundary $\partial\Omega$. Hint for 2b: divergence theorem on exercise sheet 1.

3. Let $u \in C^1(\mathbb{R}^3; \mathbb{R})$ be a real valued function. And let

$$E(u) = \int_\Omega \mathcal{L}(u(x), \nabla u(x))\,\mathrm{dx}$$

be a real valued Gâteaux differentiable functional. Calculate the Gâteaux derivative of $E(u)$. Hint: Use multidimensional integration by parts as derived in question 2b.

4. Let $u \in C^1(\Omega; \mathbb{R})$ be a real valued function with $\Omega \subset \mathbb{R}^2$. Derive the Euler-Lagrange Equation of the following energies:

   (a) The *total variation* of the function u

$$E_1(u) = \int_\Omega |\nabla u(x)|\,\mathrm{dx}.$$

   (b) The anisotropic total variation

$$E_2(u) = \int_\Omega \sqrt{\nabla u(x)^T D(x) \nabla u(x)}\,\mathrm{dx},$$

   where $D(x) \in \mathbb{R}^{2\times 2}$ is a real valued matrix.

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. In the lecture you encountered the continuous counterpart of the denoising energy from the previous exercise sheet

$$\frac{\lambda}{2}\int_\Omega (u-f)^2 \,\mathrm{dx} + \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \,\mathrm{dx},$$

where $\Omega \subset \mathbb{R}^2$ represents the image domain, $u : \Omega \to \mathbb{R}$ denotes the optimization variable and $f : \Omega \to \mathbb{R}$ stands for the input image. The corresponding Euler-Lagrange equation is given as

$$\lambda(u-f) - \Delta u = 0. \tag{1}$$

In the finite-dimensional setting, consider the images as vectors $u \in \mathbb{R}^N$, $f \in \mathbb{R}^N$.

(a) Discretize the continuous gradient operator as $\nabla = (\partial_x^+ \;\; \partial_y^+)^\mathsf{T} \in \mathbb{R}^{2N\times N}$ using forward differences with Neumann boundary conditions, representing it as a sparse matrix (`help spdiags`, `help speye`).

(b) Solve the linear system arising from (1) using the backslash command (`help \`). Note that the divergence is the negative adjoint of the gradient, thus $\Delta = \operatorname{div} \nabla = -\nabla^\mathsf{T}\nabla$.