Variational Methods for Computer Vision: Solution Sheet 1

Exercise: 24 October 2016

Part I: Theory

1. **Refresher: Multivariate analysis.**

   (a) i. $\nabla f = (x, y)^\top$

   ii. $\nabla f = (x^2 + y^2)^{-1/2}(x, y)^\top$

   (b) i. $J = \begin{pmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cos(\varphi) \end{pmatrix}$

   ii. $J = \begin{pmatrix} -r\sin(t) \\ r\cos(t) \end{pmatrix}$

   (c) i. $\operatorname{div} f = 0$

   ii. $\operatorname{div} f = 2$

   (d) The solutions for the two functions from 1c are:

   i. $\operatorname{curl} f = 2$,

   ii. $\operatorname{curl} f = 0$.

   Proof for the curl of the gradient:

$$\begin{aligned}
\operatorname{curl}(\nabla f) &= \operatorname{curl} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \\
&= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y} && \text{(Symmetry of partial derivatives)} \\
&= 0.
\end{aligned}$$

   (e) i. Using the coordinate transformation from 1(b)i with $\det J = r$, the area of a disk D_R of radius R can be calculated as

$$\begin{aligned}
\iint_{D_R} \mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi}\int_0^R r\,\mathrm{d}r\,\mathrm{d}\varphi \\
&= 2\pi\left[\frac{1}{2}r^2\right]_0^R \\
&= \pi R^2.
\end{aligned}$$

   ii. Using a parametrization like in 1(b)ii, $\gamma_R\colon [0, 2\pi] \to \mathbb{R}^2$, $f(t) = (R\cos(t), R\sin(t))^\top$ with $\|\gamma_R'\|_2 = R$, the circumference of a circle with radius R can be calculated as

$$\begin{aligned}
\int_{\gamma_R} \mathrm{d}s &= \int_0^{2\pi} R\,\mathrm{d}\varphi \\
&= 2\pi R.
\end{aligned}$$

(f) First calculate the left-hand side of the divergence theorem:

$$\begin{aligned}\iint_{D_R} \operatorname{div} f \, \mathrm{d}x \, \mathrm{d}y &= \iint_{D_R} 2 \, \mathrm{d}x \, \mathrm{d}y \\ &= 2\pi R^2. && \text{(Using 1(e)i)}\end{aligned}$$

For the right-hand side, first calculate the normal vector. The points on the boundary ∂D_R can be characterized by the zero set of $g(x, y) = x^2 + y^2 - R^2$. Calculating the gradient $\nabla g = (2x, 2y)^\top$ will give the direction of the normal n, and normalizing the gradient yields $n = (x^2 + y^2)^{-1/2}(x, y)^\top = (x, y)^\top / R$. Now the integral becomes

$$\begin{aligned}\int_{\partial D_R} \langle f, n \rangle \, \mathrm{d}s &= \int_{\gamma_R} \frac{1}{R}(x^2 + y^2) \, \mathrm{d}s \\ &= \int_{\gamma_R} R \, \mathrm{d}s \\ &= 2\pi R^2, && \text{(Using 1(e)ii)}\end{aligned}$$

which is equal to the left-hand side.

2. (a) i. Associativity:

$$\begin{aligned}((f * g) * h)(u) &= \int_{\mathbb{R}} (f * g)(x) \, h(u - x) \, \mathrm{d}x \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y) g(x - y) \, \mathrm{d}y \right) h(u - x) \, \mathrm{d}x \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) g(x - y) h(u - x) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) g(x - y) h(u - x) \, \mathrm{d}x \, \mathrm{d}y && \text{(Fubini's theorem)} \\ &= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(x - y) h(u - x) \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g((x + y) - y) h(u - (x + y)) \, \mathrm{d}x \, \mathrm{d}y && \text{(Translation invariance)} \\ &= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(x) h((u - y) - x) \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_{\mathbb{R}} f(y) (g * h)(u - y) \, \mathrm{d}y \\ &= (f * (g * h))(u).\end{aligned}$$

ii. Commutativity:

$$\begin{aligned}(f * g)(u) &:= \int_{\mathbb{R}} f(x) \, g(u - x) \, \mathrm{d}x \\ &= \int_{\varphi_u(\mathbb{R})} g(\varphi_u(x)) f(\varphi_u(u - x)) |\det J_{\varphi_u}| \, \mathrm{d}x, \\ &= \int_{\mathbb{R}} f(u - x) \, g(x) \, \mathrm{d}x && \text{with } \varphi_u(x) = u - x, |\det J_{\varphi_u}| = 1, \varphi_u(\mathbb{R}) = \mathbb{R}, \\ &= \int_{\mathbb{R}} g(x) f(u - x) \, \mathrm{d}x, \\ &=: (g * f)(u).\end{aligned}$$

iii. Distributivity:

$$\begin{aligned}
f * (g+h)(u) &= \int_{\mathbb{R}} f(x)(g+h)(u-x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} f(x)g(u-x) + f(x)h(u-x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} f(x)g(u-x)\,\mathrm{d}x + \int_{\mathbb{R}} f(x)h(u-x)\,\mathrm{d}x \\
&= (f*g + f*h)(u).
\end{aligned}$$

(b) We start with the definition of the Fourier transform:

$$\begin{aligned}
\mathcal{F}\{f*g\}(\nu) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y\right) e^{-2\pi i x\nu}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(x-y)e^{-2\pi i x\nu}\,\mathrm{d}x\right) \mathrm{d}y.
\end{aligned}$$

Introducing the substitution $z = x - y$, $\mathrm{d}z = \mathrm{d}x$ we arrive at

$$\begin{aligned}
\int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(x-y)e^{-2\pi i x\nu}\,\mathrm{d}x\right) \mathrm{d}y &= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(z)e^{-2\pi i (z+y)\nu}\,\mathrm{d}z\right) \mathrm{d}y \\
&= \int_{\mathbb{R}} f(y)e^{-2\pi i y\nu} \int_{\mathbb{R}} g(z)e^{-2\pi i z\nu}\,\mathrm{d}z\,\mathrm{d}y \\
&= \underbrace{\int_{\mathbb{R}} f(y)e^{-2\pi i y\nu}\,\mathrm{d}y}_{=:\mathcal{F}\{f\}(\nu)} \underbrace{\int_{\mathbb{R}} g(z)e^{-2\pi i z\nu}\,\mathrm{d}z}_{=:\mathcal{F}\{g\}(\nu)}.
\end{aligned}$$

As the Fourier transform can be implemented to run in $\mathcal{O}(n \log n)$ time, convolutions can be computed efficiently by exploiting this property:

$$f * g = \mathcal{F}^{-1}\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\}.$$

(c) Let us consider the difference quotient

$$\frac{(f*g)(x+t) - (f*g)(x)}{t} = \int_{\mathbb{R}} f(y)\frac{g(x+t-y) - g(x-y)}{t}\,\mathrm{d}y.$$

Now taking the limit $t \to 0$ we have

$$\begin{aligned}
\frac{d}{dx}(f*g)(x) &= \lim_{t\to 0} \frac{(f*g)(x+t) - (f*g)(x)}{t} \\
&= \lim_{t\to 0} \int_{\mathbb{R}} f(y)\frac{g(x+t-y) - g(x-y)}{t}\,\mathrm{d}y \\
&= \int_{\mathbb{R}} \lim_{t\to 0} f(y)\frac{g(x+t-y) - g(x-y)}{t}\,\mathrm{d}y \\
&= \int_{\mathbb{R}} f(y)(\frac{d}{dx}g)(x-y)\,\mathrm{d}y \\
&= f * \frac{dg}{dx} = \frac{dg}{dx} * f.
\end{aligned}$$

Remark: In order to interchange integration and limit, one needs some additional conditions to hold (see Lebesgue's dominated convergence theorem). The theorem requires that

$$F_t(y) := f(y)\frac{g(x+t-y) - g(x-y)}{t},$$

convergences pointwise to a function $F_t(y) \to F(y)$, and F_t is dominated by an integrable function g in the sense

$$|F_t(y)| \leq g(y), \forall t, \forall y.$$

The remaining equality follows from the above and commutativity of convolution:

$$\frac{d}{dx}(f * g) = \frac{d}{dx}(g * f) = g * \frac{df}{dx} = \frac{df}{dx} * g.$$