Variational Methods for Computer Vision: Solution Sheet 1

Exercise: 24 October 2016

Part I: Theory

1. **Refresher: Multivariate analysis.**

   (a) i. $\nabla f = (x, y)^\top$

   ii. $\nabla f = (x^2 + y^2)^{-1/2}(x, y)^\top$

   (b) i. $J = \begin{pmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cos(\varphi) \end{pmatrix}$

   ii. $J = \begin{pmatrix} -r\sin(t) \\ r\cos(t) \end{pmatrix}$

   (c) i. $\operatorname{div} f = 0$

   ii. $\operatorname{div} f = 2$

   (d) The solutions for the two functions from 1c are:

   i. $\operatorname{curl} f = 2$,

   ii. $\operatorname{curl} f = 0$.

   Proof for the curl of the gradient:

$$
\begin{aligned}
\operatorname{curl}(\nabla f) &= \operatorname{curl} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \\
&= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y} && \text{(Symmetry of partial derivatives)} \\
&= 0.
\end{aligned}
$$

   (e) i. Using the coordinate transformation from 1(b)i with $\det J = r$, the area of a disk D_R of radius R can be calculated as

$$
\begin{aligned}
\iint_{D_R} \mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi} \int_0^R r\,\mathrm{d}r\,\mathrm{d}\varphi \\
&= 2\pi \left[\frac{1}{2} r^2\right]_0^R \\
&= \pi R^2.
\end{aligned}
$$

   ii. Using a parametrization like in 1(b)ii, $\gamma_R \colon [0, 2\pi] \to \mathbb{R}^2$, $f(t) = (R\cos(t), R\sin(t))^\top$ with $\|\gamma_R'\|_2 = R$, the circumference of a circle with radius R can be calculated as

$$
\begin{aligned}
\int_{\gamma_R} \mathrm{d}s &= \int_0^{2\pi} R\,\mathrm{d}\varphi \\
&= 2\pi R.
\end{aligned}
$$

(f) First calculate the left-hand side of the divergence theorem:

$$\begin{aligned}\iint_{D_R} \operatorname{div} f \,\mathrm{d}x\,\mathrm{d}y &= \iint_{D_R} 2\,\mathrm{d}x\,\mathrm{d}y \\ &= 2\pi R^2. && \text{(Using 1(e)i)}\end{aligned}$$

For the right-hand side, first calculate the normal vector. The points on the boundary ∂D_R can be characterized by the zero set of $g(x,y) = x^2 + y^2 - R^2$. Calculating the gradient $\nabla g = (2x, 2y)^\top$ will give the direction of the normal n, and normalizing the gradient yields $n = (x^2+y^2)^{-1/2}(x,y)^\top = (x,y)^\top/R$. Now the integral becomes

$$\begin{aligned}\int_{\partial D_R} \langle f, n\rangle \,\mathrm{d}s &= \int_{\gamma_R} x^2 + y^2 \,\mathrm{d}s \\ &= \int_{\gamma_R} R\,\mathrm{d}s \\ &= 2\pi R^2, && \text{(Using 1(e)ii)}\end{aligned}$$

which is equal to the left-hand side.

2. (a) i. Associativity:

$$\begin{aligned}((f*g)*h)(u) &= \int_{\mathbb{R}} (f*g)(x)\,h(u-x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y\right) h(u-x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)g(x-y)h(u-x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)g(x-y)h(u-x)\,\mathrm{d}x\,\mathrm{d}y && \text{(Fubini's theorem)} \\ &= \int_{\mathbb{R}} f(y)\int_{\mathbb{R}} g(x-y)h(u-x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}} f(y)\int_{\mathbb{R}} g((x+y)-y)h(u-(x+y))\,\mathrm{d}x\,\mathrm{d}y && \text{(Translation invariance)} \\ &= \int_{\mathbb{R}} f(y)\int_{\mathbb{R}} g(x)h(u-y-x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}} f(y)(g*h)(u-y)\,\mathrm{d}y \\ &= (f*(g*h))(u).\end{aligned}$$

Remark: The translation invariance step can be seen as a special case of

$$\int_S f(s)\,\mathrm{d}s = \int_P f(\phi(p))|\det J_\phi(p)|\,\mathrm{d}p,$$

in the following way:

$$\begin{aligned}\int_{\mathbb{R}} g(x-y)\cdot h(u-x)\,\mathrm{d}x &= \int_{\mathbb{R}} (g\circ\varphi_1^y)(x)\cdot(h\circ\varphi_2^u)(x)\,\mathrm{d}x \\ &= \int_{\varphi_y(\mathbb{R})} (g\circ\varphi_1^y)(\varphi_y(x))\cdot(h\circ\varphi_2^u)(\varphi_y(x))\,\underbrace{|\det J_{\varphi_y}|}_{=1}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} g(x)h(u-y-x)\mathrm{d}x,\end{aligned}$$

with $\varphi_y(x) = x+y$, $\varphi_1^y(x) = x-y$, $\varphi_2^u(x) = u-x$.

ii. Commutativity:

$$
\begin{aligned}
(f * g)(u) &:= \int_{\mathbb{R}} f(x)\, g(u-x)\, \mathrm{d}x \\
&= \int_{\varphi_u(\mathbb{R})} g(\varphi_u(x)) f(u-\varphi_u(x)) |\det J_{\varphi_u}|\, \mathrm{d}x, \\
&= \int_{\mathbb{R}} f(u-x)\, g(x)\, \mathrm{d}x \qquad\qquad \text{with } \varphi_u(x) = u - x, |\det J_{\varphi_u}| = 1, \varphi_u(\mathbb{R}) = \mathbb{R}, \\
&= \int_{\mathbb{R}} g(x) f(u-x)\, \mathrm{d}x, \\
&=: (g * f)(u).
\end{aligned}
$$

iii. Distributivity:

$$
\begin{aligned}
f * (g+h)(u) &= \int_{\mathbb{R}} f(x)(g+h)(u-x)\, \mathrm{d}x \\
&= \int_{\mathbb{R}} f(x)g(u-x) + f(x)h(u-x)\, \mathrm{d}x \\
&= \int_{\mathbb{R}} f(x)g(u-x)\, \mathrm{d}x + \int_{\mathbb{R}} f(x)h(u-x)\, \mathrm{d}x \\
&= (f * g + f * h)(u).
\end{aligned}
$$

(b) We start with the definition of the Fourier transform:

$$
\begin{aligned}
\mathcal{F}\{f * g\}(\nu) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y) g(x-y)\, \mathrm{d}y \right) e^{-2\pi i x \nu}\, \mathrm{d}x \\
&= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(x-y) e^{-2\pi i x \nu}\, \mathrm{d}x \right) \mathrm{d}y.
\end{aligned}
$$

Introducing the substitution $z = x - y$, $\mathrm{d}z = \mathrm{d}x$ we arrive at

$$
\begin{aligned}
\int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(x-y) e^{-2\pi i x \nu}\, \mathrm{d}x \right) \mathrm{d}y &= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(z) e^{-2\pi i (z+y) \nu}\, \mathrm{d}z \right) \mathrm{d}y \\
&= \int_{\mathbb{R}} f(y) e^{-2\pi i y \nu} \int_{\mathbb{R}} g(z) e^{-2\pi i z \nu}\, \mathrm{d}z\, \mathrm{d}y \\
&= \underbrace{\int_{\mathbb{R}} f(y) e^{-2\pi i y \nu}\, \mathrm{d}y}_{=:\mathcal{F}\{f\}(\nu)} \; \underbrace{\int_{\mathbb{R}} g(z) e^{-2\pi i z \nu}\, \mathrm{d}z}_{=:\mathcal{F}\{g\}(\nu)}.
\end{aligned}
$$

As the Fourier transform can be implemented to run in $\mathcal{O}(n \log n)$ time, convolutions can be computed efficiently by exploiting this property:

$$
f * g = \mathcal{F}^{-1}\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\}.
$$

(c) Let us consider the difference quotient

$$
\frac{(f * g)(x+t) - (f * g)(x)}{t} = \int_{\mathbb{R}} f(y) \frac{g(x+t-y) - g(x-y)}{t}\, \mathrm{d}y.
$$

Now taking the limit $t \to 0$ we have

$$\begin{aligned}
\frac{d}{dx}(f * g)(x) &= \lim_{t\to 0} \frac{(f * g)(x+t) - (f * g)(x)}{t} \\
&= \lim_{t\to 0} \int_{\mathbb{R}} f(y) \frac{g(x+t-y) - g(x-y)}{t} \,\mathrm{d}y \\
&= \int_{\mathbb{R}} \lim_{t\to 0} f(y) \frac{g(x+t-y) - g(x-y)}{t} \,\mathrm{d}y \\
&= \int_{\mathbb{R}} f(y) (\frac{d}{dx} g)(x-y) \,\mathrm{d}y \\
&= f * \frac{dg}{dx} = \frac{dg}{dx} * f.
\end{aligned}$$

Remark: In order to interchange integration and limit, one needs some additional conditions to hold (see Lebesgue's dominated convergence theorem). The theorem requires that

$$F_t(y) := f(y) \frac{g(x+t-y) - g(x-y)}{t},$$

convergences pointwise to a function $F_t(y) \to F(y)$, and F_t is dominated by an integrable function g in the sense

$$|F_t(y)| \leq g(y), \forall t, \forall y.$$

The remaining equality follows from the above and commutativity of convolution:

$$\frac{d}{dx}(f * g) = \frac{d}{dx}(g * f) = g * \frac{df}{dx} = \frac{df}{dx} * g.$$