Variational Methods for Computer Vision: Solution Sheet 3

Exercise: November 7, 2016

Part I: Theory

1. (a) Suppose x^* is a local but not a global minimizer. Then there exists a $z \in \mathbb{R}^n$ with $f(z) < f(x^*)$. Consider the line segment

$$x_\lambda = \lambda z + (1-\lambda)x^*, \lambda \in (0,1).$$

By convexity we have:

$$f(x_\lambda) = f(\lambda z + (1-\lambda)x^*) \leq \lambda f(z) + (1-\lambda)f(x^*) < \lambda f(x^*) + (1-\lambda)f(x^*) = f(x^*).$$

$\Rightarrow$ Any neighbourhood of x^* contains a point x_λ with $f(x_\lambda) < f(x^*)$, which is a contradiction to the assumption.

(b) Assume that x^* is a stationary point but not a global minimizer. Then there is a $z \in \mathbb{R}^n$ with $f(z) < f(x^*)$, and

$$\begin{aligned} \langle \nabla f(x^*), z - x^* \rangle &= \lim_{\epsilon \to 0} \frac{1}{\varepsilon}(f(x^* + \varepsilon(z - x^*)) - f(x^*)) \\ &\leq \lim_{\epsilon \to 0} \frac{1}{\varepsilon}(\varepsilon f(z) + (1-\varepsilon)f(x^*) - f(x^*)) \\ &= f(z) - f(x^*) < 0. \end{aligned} \tag{1}$$

Thus $\langle \nabla f(x^*), z - x^* \rangle \neq 0 \Rightarrow \nabla f(x^*) \neq 0 \Rightarrow x^*$ is not a stationary point.

2. <u>f convex $\Rightarrow$ (epi f) convex:</u>

Take (u,a), $(v,b) \in$ epi f. Then

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda)f(v) \tag{2}$$
$$\leq \lambda a + (1-\lambda)b \tag{3}$$

Thus $(\lambda u + (1-\lambda)v, \lambda a + (1-\lambda)b) = \lambda(u,a) + (1-\lambda)(v,b) \in$ epi f.

<u>(epi f) convex $\Rightarrow$ f convex:</u>

Let $a = f(x), b = f(y)$. Then $(x,a), (y,b) \in$ epi f. Since epi f is convex:

$$(\lambda x + (1-\lambda)y, \lambda a + (1-\lambda)b) \in \text{epi } f.$$

Thus we have convexity of f:

$$f(\lambda x + (1-\lambda)y) \leq \lambda a + (1-\lambda)b = \lambda f(x) + (1-\lambda)f(y).$$

3. (a) A direct calculation shows:

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &= \alpha f(\lambda x + (1-\lambda)y) + \beta g(\lambda x + (1-\lambda)y) \\ &\leq \alpha\lambda f(x) + \alpha(1-\lambda)f(y) + \beta\lambda g(x) + \beta(1-\lambda)g(y) \\ &= \lambda(\alpha f(x) + \beta g(x)) + (1-\lambda)(\alpha f(y) + \beta g(y)) \\ &= \lambda h(x) + (1-\lambda)h(y). \end{aligned} \tag{4}$$

(b) We see that

$$\begin{aligned} \text{epi } f \cap \text{epi } g &= \{(x,a) \mid f(x) \le a\} \cap \{(x,a) \mid g(x) \le a\} \\ &= \{(x,a) \mid \max\{f(x), g(x)\} \le a\} = \text{epi } h \end{aligned} \tag{5}$$

Since the intersection of two convex sets is always convex, epi h is a convex set. This implies by the exercise 2 that h is also a convex function.

Proof that the intersection of two convex sets is convex (always $\alpha \in (0,1)$):

$$\begin{aligned} S_1, S_2 \text{ convex} \Rightarrow \quad & (\forall x, y \in S_1 \colon \alpha x + (1-\alpha) y \in S_1) & (6)\\ & \wedge (\forall x, y \in S_2 \colon \alpha x + (1-\alpha) y \in S_2) & (7)\\ \Rightarrow \quad & (x, y \in S_1 \wedge x, y \in S_2 & (8)\\ & \Rightarrow \alpha x + (1-\alpha) y \in S_1 \wedge \alpha x + (1-\alpha) y \in S_2) & (9)\\ \Rightarrow \quad & \forall x, y \in S_1 \cap S_2 \colon \alpha x + (1-\alpha) y \in S_1 \cap S_2 & (10)\\ \Rightarrow \quad & S_1 \cap S_2 \text{ convex.} & (11) \end{aligned}$$

(c) Counterexample: $h(x) = \min\{(x-1)^2, (x+1)^2\}$ is clearly not convex.

4.

$$\begin{aligned} h''(x) &= f(g(x))'' = \big(f'(g(x)) g'(x)\big)' \\ &= \underbrace{f''(g(x))}_{\ge 0} \underbrace{g'(x) g'(x)}_{\ge 0} + f'(g(x)) \underbrace{g''(x)}_{\ge 0} \end{aligned} \tag{12}$$

Thus $h''(x) \ge 0$ if $f'(g(x)) \ge 0$. Hence f has to be a convex non-decreasing function.