Variational Methods for Computer Vision: Solution Sheet 4

Exercise: November 14, 2014

Part I: Theory

1. We proceed accordingly to the lecture:

$$
\begin{aligned}
\frac{\partial E(u)}{\partial u}\Big|_h &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(E(u+\varepsilon h)-E(u)\right)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^b L(u+\varepsilon h, u'+\varepsilon h', u''+\varepsilon h'') - L(u,u',u'')\,\mathrm{d}x\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^b L(u,u',u'') + \frac{\partial L}{\partial u}\varepsilon h + \frac{\partial L}{\partial u'}\varepsilon h' + \frac{\partial L}{\partial u''}\varepsilon h'' + \mathcal{O}(\varepsilon^2) - L(u,u',u'')\,\mathrm{d}x\\
&= \int_a^b \frac{\partial L}{\partial u}h + \frac{\partial L}{\partial u'}h' + \frac{\partial L}{\partial u''}h''\,\mathrm{d}x\\
&= \int_a^b \frac{\partial L}{\partial u}h\,\mathrm{d}x - \int_a^b h\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'}\,\mathrm{d}x - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u''}h'\,\mathrm{d}x + \left[h\frac{\partial L}{\partial u'}\right]_a^b + \left[h'\frac{\partial L}{\partial u''}\right]_a^b\\
&= \int_a^b h\left(\frac{\partial L}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial L}{\partial u''}\right)\,\mathrm{d}x - \left[h\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u''}\right]_a^b + \left[h\frac{\partial L}{\partial u'}\right]_a^b + \left[h'\frac{\partial L}{\partial u''}\right]_a^b.
\end{aligned}
$$

2. (a)

$$
\begin{aligned}
\operatorname{div}(fg) &= \sum_{i=1}^n \frac{\partial(fg_i)}{\partial x_i} = \sum_{i=1}^n\left(\frac{\partial f}{\partial x_i}g_i + f\frac{\partial g_i}{\partial x_i}\right) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}g_i + f\sum_{i=1}^n\frac{\partial g_i}{\partial x_i}\\
&= \nabla f\cdot g + f\operatorname{div} g.
\end{aligned}
$$

(b) By the divergence theorem, we know that $\int_\Omega \operatorname{div}(fg)\,\mathrm{d}x = \int_{\partial\Omega}\langle fg, n\rangle\,\mathrm{d}s$. Replacing the divergence with the result from question 2a yields

$$
\int_\Omega \nabla f\cdot g\,\mathrm{d}x + \int_\Omega f\operatorname{div} g\,\mathrm{d}x = \int_{\partial\Omega} f\langle g,n\rangle\,\mathrm{d}s,
$$

which is easily transformed into the desired result,

$$
\int_\Omega \nabla f\cdot g\,\mathrm{d}x = \int_{\partial\Omega} f\langle g,n\rangle\,\mathrm{d}s - \int_\Omega f\operatorname{div} g\,\mathrm{d}x.
$$

3.

$$
\begin{aligned}
\left.\frac{\partial E(u)}{\partial u}\right|_h &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(E(u+\varepsilon h)-E(u)\right)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega L(u+\varepsilon h,\nabla(u+\varepsilon h)) - L(u,\nabla u)\,\mathrm{d}x\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega L(u,\nabla u)+\frac{\partial L}{\partial u}\varepsilon h+\frac{\partial L}{\partial u_x}\varepsilon\frac{\partial h}{\partial x}+\frac{\partial L}{\partial u_y}\varepsilon\frac{\partial h}{\partial y}+\frac{\partial L}{\partial u_z}\varepsilon\frac{\partial h}{\partial z}+\mathcal{O}(\varepsilon^2)-L(u,\nabla u)\,\mathrm{d}x\\
&= \int_\Omega \frac{\partial L}{\partial u}h\,\mathrm{d}x+\int_\Omega\left\langle \nabla h,\left(\frac{\partial L}{\partial u_x}\;\frac{\partial L}{\partial u_y}\;\frac{\partial L}{\partial u_z}\right)^T\right\rangle\mathrm{d}x\\
&= \int_\Omega \frac{\partial L}{\partial u}h\,\mathrm{d}x-\int_\Omega h\,\mathrm{div}\left(\frac{\partial L}{\partial u_x}\;\frac{\partial L}{\partial u_y}\;\frac{\partial L}{\partial u_z}\right)^T\mathrm{d}x\\
&\qquad+\int_{\partial\Omega} h\left\langle\left(\frac{\partial L}{\partial u_x}\;\frac{\partial L}{\partial u_y}\;\frac{\partial L}{\partial u_z}\right)^T,n\right\rangle\mathrm{d}\partial\Omega\\
&= \int_\Omega h\left(\frac{\partial L}{\partial u}-\left(\frac{\partial}{\partial x}\frac{\partial L}{\partial u_x}+\frac{\partial}{\partial y}\frac{\partial L}{\partial u_y}+\frac{\partial}{\partial z}\frac{\partial L}{\partial u_z}\right)\right)\mathrm{d}x\\
&\qquad+\int_{\partial\Omega} h\left(\frac{\partial L}{\partial u_x}n_x+\frac{\partial L}{\partial u_y}n_y+\frac{\partial L}{\partial u_z}n_z\right)\mathrm{d}\partial\Omega.
\end{aligned}
$$

Hence we can write the Euler-Lagrange equation as

$$
\begin{aligned}
\frac{\partial L}{\partial u}-\mathrm{div}\left(\frac{\partial L}{\partial\nabla u}\right)&=0, &&\text{on } \Omega,\\
\left\langle\frac{\partial L}{\partial\nabla u},\nu\right\rangle&=0, &&\text{on } \partial\Omega,
\end{aligned}
$$

where ν denotes the normal vector on the boundary $\partial\Omega$.

4. (a) We have that $L(u,\nabla u)=\sqrt{u_x^2+u_y^2}$ and

$$
\begin{aligned}
\frac{\partial L}{\partial u_x}&=\frac{u_x}{\sqrt{u_x^2+u_y^2}},\\
\frac{\partial L}{\partial u_y}&=\frac{u_y}{\sqrt{u_x^2+u_y^2}}.
\end{aligned}
$$

Thus the Euler-Lagrange equation is given as the following:

$$\frac{\partial E}{\partial u}=\frac{\partial L}{\partial u}-\mathrm{div}\left(\frac{\partial L}{\partial\nabla u}\right)=-\,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

(b) Similarly to the previous exercise we arrive for $L(u,\nabla u)=\sqrt{(\nabla u)^\mathsf{T}D\nabla u}$ at the following Euler-Lagrange equation:

$$\frac{\partial E}{\partial u}=\frac{\partial L}{\partial u}-\mathrm{div}\left(\frac{\partial L}{\partial\nabla u}\right)=-\,\mathrm{div}\left(\frac{(D+D^\mathsf{T})\nabla u}{2\sqrt{(\nabla u)^\mathsf{T}D\nabla u}}\right).$$