Variational Methods for Computer Vision: Solution Sheet 5

Exercise: November 21, 2016

Part I: Theory

1. Consider the functional involving first-order and second-order derivates

$$E(u) = \int_\Omega \mathcal{L}(u, \nabla u, \nabla^2 u)\,\mathrm{d}x.$$

We compute the Euler-Lagrange equation, starting from the directional (Gateaux) derivative:

$$\begin{aligned}
\left.\frac{dE(u)}{du}\right|_h &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(E(u+\varepsilon h) - E(u)\right)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \left(\mathcal{L}(u+\varepsilon h, \nabla(u+\varepsilon h), \nabla^2(u+\varepsilon h)) - \mathcal{L}(u,\nabla u, \nabla^2 u\right)\,\mathrm{d}x\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \left(\mathcal{L}(u+\varepsilon h, \nabla u+\varepsilon\nabla h, \nabla^2 u+\varepsilon\nabla^2 h) - \mathcal{L}(u,\nabla u, \nabla^2 u)\right)\,\mathrm{d}x\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \mathcal{L}(u,\nabla u,\nabla^2 u) + \frac{\partial\mathcal{L}}{\partial u}\varepsilon h + \langle\frac{\partial\mathcal{L}}{\partial\nabla u},\varepsilon\nabla h\rangle + \langle\frac{\partial\mathcal{L}}{\partial\nabla^2 u},\varepsilon\nabla^2 h\rangle - \mathcal{L}(u,\nabla u,\nabla^2 u)\,\mathrm{d}x\\
&= \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial u}h + \langle\frac{\partial\mathcal{L}}{\partial\nabla u},\nabla h\rangle + \langle\frac{\partial\mathcal{L}}{\partial\nabla^2 u},\nabla^2 h\rangle\right)\,\mathrm{d}x\\
&= \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial u}h - \operatorname{div}\frac{\partial\mathcal{L}}{\partial\nabla u}h + \langle\frac{\partial\mathcal{L}}{\partial\nabla^2 u},\nabla^2 h\rangle\right)\,\mathrm{d}x + \int_{\partial\Omega} h\langle\frac{\partial\mathcal{L}}{\partial\nabla u},\nu\rangle\mathrm{d}s
\end{aligned}$$

For the term involving the Hessian, we have

$$\begin{aligned}
\int_\Omega \frac{\partial\mathcal{L}}{\partial\nabla^2 u}\nabla^2 h\mathrm{d}x &= \int_\Omega \frac{\partial\mathcal{L}}{\partial\nabla^2 u}\nabla\cdot\nabla h\mathrm{d}x\\
&= \int_\Omega -\operatorname{div}\frac{\partial\mathcal{L}}{\partial\nabla^2 u}\nabla h\mathrm{d}x + \int_{\partial\Omega}\nabla h\cdot\langle\frac{\partial\mathcal{L}}{\partial\nabla^2 u},\nu\rangle\mathrm{d}s\\
&= \int_\Omega \operatorname{div}^2\frac{\partial\mathcal{L}}{\partial\nabla^2 u}h\mathrm{d}x + \int_{\partial\Omega}\nabla h\cdot\langle\frac{\partial\mathcal{L}}{\partial\nabla^2 u},\nu\rangle\mathrm{d}s - \int_{\partial\Omega} h\cdot\langle\operatorname{div}\frac{\partial\mathcal{L}}{\partial\nabla^2 u},\nu\rangle\mathrm{d}s
\end{aligned}$$

Assuming Neumann boundary conditions, we only consider variations h which also have vanishing gradient on the boundary $\partial\Omega$, hence the term $\int_{\partial\Omega}\nabla h\cdot\langle\frac{\partial\mathcal{L}}{\partial\nabla^2 u},\nu\rangle\mathrm{d}s$ is zero. Inserting this into the first calculation leads to

$$... = \int_\Omega h\left(\frac{\partial\mathcal{L}}{\partial u} - \operatorname{div}\frac{\partial\mathcal{L}}{\partial\nabla u} + \operatorname{div}^2\frac{\partial\mathcal{L}}{\partial\nabla^2 u}\right)\,\mathrm{d}x + \int_{\partial\Omega} h\langle\frac{\partial\mathcal{L}}{\partial\nabla u} - \operatorname{div}\frac{\partial\mathcal{L}}{\partial\nabla^2 u},\nu\rangle\mathrm{d}s$$

Hence we arrive at the Euler-Lagrange equations:

$$\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial u} - \operatorname{div}\frac{\partial\mathcal{L}}{\partial\nabla u} + \operatorname{div}^2\frac{\partial\mathcal{L}}{\partial\nabla^2 u} = 0,\ \text{for all } x\in\Omega,\\
&\langle\frac{\partial\mathcal{L}}{\partial\nabla u} - \operatorname{div}\frac{\partial\mathcal{L}}{\partial\nabla^2 u},\nu\rangle = 0,\ \text{for all } x\in\partial\Omega, \nu \text{ is normal at point } x.
\end{aligned}$$

2. In the following, we calculate the Euler-Lagrange equation of the general functional

$$E(u) = \int_\Omega \mathcal{L}(u, \nabla u, u * k)\,\mathrm{d}x.$$

For that we consider the directional derivative

$$\begin{aligned}
\left.\frac{dE(u)}{du}\right|_h &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(E(u+\varepsilon h) - E(u)\right) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega \left(\mathcal{L}(u+\varepsilon h, \nabla(u+\varepsilon h), (u+\varepsilon h) * k) - \mathcal{L}(u, \nabla u, u * k)\right)\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega \left(\mathcal{L}(u+\varepsilon h, \nabla u + \varepsilon \nabla h, u * k + \varepsilon (h * k)) - \mathcal{L}(u, \nabla u, u * k)\right)\,\mathrm{d}x \\
&= \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u} h + \frac{\partial \mathcal{L}}{\partial \nabla u}\nabla h + \frac{\partial \mathcal{L}}{\partial (u*k)}(h * k)\right)\,\mathrm{d}x
\end{aligned}$$

Now note that for convolution it holds that

$$\begin{aligned}
\int g(x)(u*k)(x)\,\mathrm{d}x &= \int\int u(y)k(x-y)g(x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int\int g(x)k(x-y)\,\mathrm{d}x\,u(y)\,\mathrm{d}y \\
&= \int\int g(x)\bar{k}(y-x)\,\mathrm{d}x\,u(y)\,\mathrm{d}y \\
&= \int (g * \bar{k})(y)\,u(y)\,\mathrm{d}y,
\end{aligned}$$

where $\bar{k}(x) = (k \circ -)(x) = k(-x)$.

If we consider the convolution $k * u$ as the application of a linear operator $Au = k * u$, then the above derivation shows that the so called *adjoint* operator satisfying

$$\langle g, Au\rangle = \langle A^* g, u\rangle, \qquad \forall u, g$$

is given as $A^* u = \bar{k} * u$.

Applying that, and partial integration yields

$$\begin{aligned}
&\int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u} h + \frac{\partial \mathcal{L}}{\partial \nabla u}\nabla h + \frac{\partial \mathcal{L}}{\partial (u*k)}(h * k)\right)\,\mathrm{d}x = \\
&\int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u} - \operatorname{div}\frac{\partial \mathcal{L}}{\partial \nabla u} + \frac{\partial \mathcal{L}}{\partial (u*k)} * \bar{k}\right) h\,\mathrm{d}x + \int_{\partial\Omega} h\left\langle \frac{\partial \mathcal{L}}{\partial \nabla u}, \nu\right\rangle\,\mathrm{d}\partial\Omega.
\end{aligned}$$

Thus the Euler-Lagrange equations for the functional

$$E(u) = \int_\Omega \frac{\lambda}{2}((u*k)(x) - f(x))^2 + |\nabla u(x)|\,\mathrm{d}x,$$

is given as the following:

$$\begin{aligned}
\lambda\left((u*k) - f\right) * \bar{k} - \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) &= 0, && \text{in } \Omega, \\
\left\langle \frac{\nabla u}{|\nabla u|}, \nu\right\rangle &= 0, && \text{on } \partial\Omega.
\end{aligned}$$