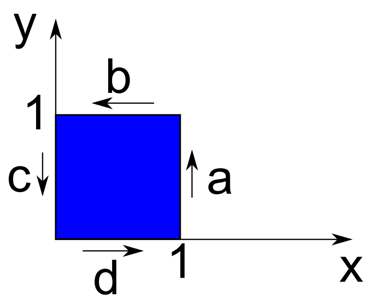
Variational Methods for Computer Vision: Solution Sheet 7

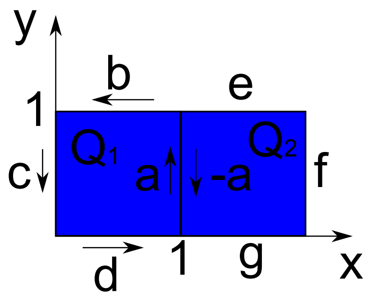
Exercise: December 5, 2016

Part I: Theory



1. (a)

$$
\begin{aligned}
\int_Q v_x(x,y) - u_y(x,y)\mathrm{d}x\mathrm{d}y &= \int_0^1 \int_0^1 v_x(x,y) - u_y(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^1 \int_0^1 v_x(x,y)\mathrm{d}x\mathrm{d}y - \int_0^1 \int_0^1 u_y(x,y)\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 v(x,y)\big|_{x=0}^{x=1}\mathrm{d}y - \int_0^1 v(x,y)\big|_{y=0}^{y=1}\mathrm{d}x \\
&= \int_0^1 v(1,y) - v(0,y)\mathrm{d}y - \int_0^1 v(x,1) - v(x,0)\mathrm{d}x \\
&= \int_0^1 v(1,y)\mathrm{d}y - \int_0^1 v(0,y)\mathrm{d}y - \int_0^1 v(x,1)\mathrm{d}x + \int_0^1 v(x,0)\mathrm{d}x \\
&= \underbrace{\int_0^1 v(1,y)\mathrm{d}y}_{\int_a V(s)d\vec{s}} + \underbrace{\int_1^0 v(0,y)\mathrm{d}y}_{\int_c V(s)d\vec{s}} + \underbrace{\int_1^0 v(x,1)\mathrm{d}x}_{\int_b V(s)d\vec{s}} + \underbrace{\int_0^1 v(x,0)\mathrm{d}x}_{\int_d V(s)d\vec{s}} \\
&= \oint_{\partial Q} V(s)d\vec{s}.
\end{aligned}
$$



(b) Start by joining two squares and using the result from 1a.

$$
\begin{aligned}
&\int_{Q_1} v_x(x,y) - u_y(x,y)\mathrm{d}x\mathrm{d}y + \int_{Q_2} v_x(x,y) - u_y(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_a V(s)d\vec{s} + \int_b V(s)d\vec{s} + \int_c V(s)d\vec{s} + \int_d V(s)d\vec{s} \\
&- \int_a V(s)d\vec{s} + \int_g V(s)d\vec{s} + \int_f V(s)d\vec{s} + \int_e V(s)d\vec{s} \\
&= \int_b V(s)d\vec{s} + \int_c V(s)d\vec{s} + \int_d V(s)d\vec{s} + \int_g V(s)d\vec{s} + \int_f V(s)d\vec{s} + \int_e V(s)d\vec{s} \\
&= \oint_{\partial(Q_1 \cup Q_2)} V(s)d\vec{s}.
\end{aligned}
$$

More squares can be added in exactly the same manner; the line integrals on the interface will always appear twice with different signs.

2. Consider the energies of regions Ω_1 and Ω_2 *before* and *after* the merge operation:

$$
\begin{aligned}
E_{\text{before}} &= \int_{\Omega_1} (I(x) - u_1)^2 \mathrm{d}x + \int_{\Omega_2} (I(x) - u_2)^2 \mathrm{d}x + \nu |C_{\text{before}}| \\
E_{\text{after}} &= \int_{\Omega_1 \cup \Omega_2} (I(x) - u_{\text{merged}})^2 \mathrm{d}x + \nu |C_{\text{after}}|.
\end{aligned}
$$

Here we assume that u_1, u_2 and u_{merged} optimize the energy given the respective region boundaries, i.e. they are the average intensity of the respective region (shown in the lecture). From this it follows that

$$
u_{\text{merged}} = \frac{u_1 A_1 + u_2 A_2}{A_1 + A_2}, \tag{1}
$$

which means u_{merged} is a weighted average of u_1 and u_2. Furthermore we are going to use the identity

$$
\int_{\Omega} (f(x) - \overline{f})^2 \mathrm{d}x = \int_{\Omega} f(x)^2 \mathrm{d}x - |\Omega| \overline{f}^2, \tag{2}
$$

which is true in particular for $f = I$ and $\overline{f} = u_i$.

So the change in energy δE becomes:

$$
\begin{aligned}
\delta E &= E_{\text{after}} - E_{\text{before}} \\
&= \int\limits_{\Omega_1 \cup \Omega_2} (I(x) - u_{\text{merged}})^2 \mathrm{d}x - \int\limits_{\Omega_1} (I(x) - u_1)^2 \mathrm{d}x - \int\limits_{\Omega_2} (I(x) - u_2)^2 \mathrm{d}x - \nu \delta C \\
&= \int\limits_{\Omega_1 \cup \Omega_2} I(x)^2 \mathrm{d}x - (A_1 + A_2) u_{\text{merged}}^2 && \text{(using (2))} \\
&- \int\limits_{\Omega_1} I(x)^2 \mathrm{d}x + A_1 {u_1}^2 - \int\limits_{\Omega_2} I(x)^2 \mathrm{d}x + A_2 {u_2}^2 - \nu \delta C \\
&= A_1 {u_1}^2 + A_2 {u_2}^2 - (A_1 + A_2) \left(\frac{u_1 A_1 + u_2 A_2}{A_1 + A_2} \right)^2 - \nu \delta C && \text{(using (1))} \\
&= A_1 {u_1}^2 + A_2 {u_2}^2 - \frac{(u_1 A_1 + u_2 A_2)^2}{A_1 + A_2} - \nu \delta C \\
&= A_1 {u_1}^2 + A_2 {u_2}^2 - \frac{(u_1 A_1)^2 + 2 u_1 A_1 u_2 A_2 + (u_2 A_2)^2}{A_1 + A_2} - \nu \delta C \\
&= \frac{(A_1 + A_2) A_1 {u_1}^2 + (A_1 + A_2) A_2 {u_2}^2 - (u_1 A_1)^2 - 2 u_1 A_1 u_2 A_2 - (u_2 A_2)^2}{A_1 + A_2} - \nu \delta C \\
&= \frac{A_1 A_2 {u_1}^2 + A_1 A_2 {u_2}^2 - 2 A_1 A_2 u_1 u_2}{A_1 + A_2} - \nu \delta C \\
&= \frac{A_1 A_2}{A_1 + A_2} (u_1 - u_2)^2 - \nu \delta C.
\end{aligned}
$$