Variational Methods for Computer Vision: Solution Sheet 8

Exercise: December 12, 2016

Part I: Theory

1. Recall from the lecture, that the Euler-Lagrange equation for the two-region Mumford-Shah functional for a curve $C : [0,1] \to \Omega \subset \mathbb{R}^2$ and image $I : \Omega \to \mathbb{R}$ are given by

$$\frac{dE}{dC} = \left((I - u_{\text{int}})^2 - (I - u_{\text{ext}})^2 + \nu\kappa\right) n_C. \tag{1}$$

Here, u_{int} and u_{ext} are the average intensities inside and outside the curve C, i.e.,

$$u_{\text{int}} = \frac{\int_{\text{int}(C)} I(x)\mathrm{d}x}{\int_{\text{int}(C)} \mathrm{d}x}, \; u_{\text{ext}} = \frac{\int_{\text{ext}(C)} I(x)\mathrm{d}x}{\int_{\text{ext}(C)} \mathrm{d}x}. \tag{2}$$

We will consider the curve evolution

$$\frac{\partial C}{\partial t} = -\frac{dE}{dC} = \left(-(I - u_{\text{int}})^2 + (I - u_{\text{ext}})^2 - \nu\kappa\right) n_C. \tag{3}$$

Intuitively, we evolve the curve along the normal vector n_C depending on the sign of the term in the brackets.

(a) The curvature κ of a circle with radius r is $\kappa = \frac{1}{r}$. We can use this fact in calculating the Euler-Lagrange equations for the 2 different cases.

Case $r > 1$:

$$u_{\text{ext}} = 0, \quad , u_{\text{int}} = \frac{\pi}{\pi r^2} = \frac{1}{r^2}.$$

This leads to following inner term:

$$(I - u_{\text{ext}})^2 - (I - u_{\text{int}})^2 - \nu\kappa = (0-0)^2 - (0 - \frac{1}{r^2})^2 - \frac{\nu}{r} = -\frac{1}{r^4} - \frac{\nu}{r}.$$

Case $r \leq 1$:

$$u_{\text{ext}} = \frac{\pi - \pi r^2}{100 - \pi r^2}, \quad u_{\text{int}} = 1.$$

A short computation shows:

$$\begin{aligned}(I - u_{\text{ext}})^2 - (I - u_{\text{int}})^2 - \nu\kappa &= \left(1 - \frac{\pi - \pi r^2}{100 - \pi r^2}\right)^2 - 0 - \frac{\nu}{r} \\ &= \left(\frac{100 - \pi}{100 - \pi r^2}\right)^2 - \frac{\nu}{r}.\end{aligned}$$

(b) We see that the limits differ,

$$\lim_{r \searrow 1} -\frac{1}{r^4} - \frac{\nu}{r} = -1 - \nu,$$
$$\lim_{r \nearrow 1} -\frac{100-\pi}{100-\pi r^2} - \frac{\nu}{r} = \frac{100-\pi}{100-\pi} - \nu = 1 - \nu,$$

hence, the Gateâux derivative at $r = 1$ is not continuous.

This shows that the original energy $E(C)$ is not differentiable, which can lead to convergence problems when using gradient descent-type algorithms as they technically require differentiability of the energy.

$\nu \leq 1$ is a good choice because it ensures that the curve evolves in the right direction for both cases $r > 1$ and $r \leq 1$.

2. Let the curve $C : [0,1] \to \mathbb{R}^2$ be given by $C(s) = \begin{bmatrix} x(s) & y(s) \end{bmatrix}^\top$, and denote the derivative as $C'(s) = \begin{bmatrix} \dot{x}(s) & \dot{y}(s) \end{bmatrix}^\top$. We rewrite the energy functional as the following

$$E(C) = \int_0^1 g(C(s)) \|C'(s)\| \mathrm{d}s = \int_0^1 L(x, y, \dot{x}, \dot{y}) \mathrm{d}s, \tag{4}$$

for a Lagrangian $L : \mathbb{R}^4 \to \mathbb{R}$, given by $L(a,b,c,d) = g(a,b)\sqrt{c^2+d^2}$.

Following the lecture, and similar to the previous exercise sheets, the Euler-Lagrange equations are given by:

$$\frac{\partial L}{\partial x}(x,y,\dot{x},\dot{y}) - \frac{d}{ds}\frac{\partial L}{\partial \dot{x}}(x,y,\dot{x},\dot{y}) = 0,$$
$$\frac{\partial L}{\partial y}(x,y,\dot{x},\dot{y}) - \frac{d}{ds}\frac{\partial L}{\partial \dot{y}}(x,y,\dot{x},\dot{y}) = 0,$$

for $x : [0,1] \to \mathbb{R}$, $y : [0,1] \to \mathbb{R}$, $\dot{x} : [0,1] \to \mathbb{R}$, $\dot{y} : [0,1] \to \mathbb{R}$, at all $s \in [0,1]$. There are no boundary terms, as we assume C to be a closed curve.

For the partial derivatives of L we have

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x} g(x,y)\sqrt{\dot{x}^2+\dot{y}^2} = \frac{\partial}{\partial x} g(x,y)\frac{\dot{x}^2+\dot{y}^2}{\sqrt{\dot{x}^2+\dot{y}^2}}$$
$$\frac{\partial L}{\partial y} = \frac{\partial}{\partial y} g(x,y)\sqrt{\dot{x}^2+\dot{y}^2} = \frac{\partial}{\partial y} g(x,y)\frac{\dot{x}^2+\dot{y}^2}{\sqrt{\dot{x}^2+\dot{y}^2}}$$
$$\frac{\partial L}{\partial \dot{x}} = g(x,y)\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$
$$\frac{\partial L}{\partial \dot{y}} = g(x,y)\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

Denote by $t = \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix} \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}$ the unit tangent vector of the curve, and by $n = \begin{bmatrix} -\dot{y} & \dot{x} \end{bmatrix} \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}$ the unit normal vector. Then it can be verified by a simple calculation that

$$\frac{d}{ds} t = \kappa n, \tag{*}$$

where κ denotes the curvature. Taking the derivative with respect to s and using the product rule, we have for the last terms:

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{x}} = (\frac{d}{ds}g(x,y))\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} + g(x,y)(\frac{d}{ds}\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}})$$
$$\frac{d}{ds}\frac{\partial L}{\partial \dot{y}} = (\frac{d}{ds}g(x,y))\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} + g(x,y)(\frac{d}{ds}\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}})$$

This can be rewritten using (*) as

$$\langle \nabla g(x,y), t\rangle \begin{bmatrix}\dot{x} & \dot{y}\end{bmatrix}^\top + g(x,y)\kappa n,$$

and the whole Euler-Lagrange equation becomes

$$\begin{aligned}
&\nabla g(x,y)\sqrt{\dot{x}^2+\dot{y}^2} - \langle \nabla g(x,y), t\rangle \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} - g(x,y)\kappa n\\
&= \begin{bmatrix}g_x(x,y)\\ g_y(x,y)\end{bmatrix}\sqrt{\dot{x}^2+\dot{y}^2} - \begin{bmatrix}\dot{x}(\dot{x}g_x(x,y)+\dot{y}g_y(x,y))\\ \dot{y}(\dot{x}g_x(x,y)+\dot{y}g_y(x,y))\end{bmatrix}\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}} - g(x,y)\kappa n\\
&= \begin{bmatrix}g_x(x,y)\\ g_y(x,y)\end{bmatrix}\frac{\dot{x}^2+\dot{y}^2}{\sqrt{\dot{x}^2+\dot{y}^2}} - \begin{bmatrix}\dot{x}(\dot{x}g_x(x,y)+\dot{y}g_y(x,y))\\ \dot{y}(\dot{x}g_x(x,y)+\dot{y}g_y(x,y))\end{bmatrix}\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}} - g(x,y)\kappa n\\
&= \begin{bmatrix}\dot{y}^2g_x(x,y) - \dot{x}\dot{y}g_y(x,y)\\ \dot{x}^2g_y(x,y) - \dot{y}\dot{x}g_x(x,y)\end{bmatrix}\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}} - g(x,y)\kappa n\\
&= \langle \nabla g(x,y), n\rangle n\sqrt{\dot{x}^2+\dot{y}^2} - g(x,y)\kappa n
\end{aligned}$$

Assuming an arc-length parametrization of the curve, i.e., $\sqrt{\dot{x}^2+\dot{y}^2}=1$, we arrive at

$$\frac{dE}{dC} = (\langle \nabla g, n\rangle - g\kappa)n.$$

This leads to the final gradient descent curve evolution

$$\frac{\partial C}{\partial t} = -\frac{dE}{dC} = (g\kappa - \langle \nabla g, n\rangle)n.$$

Remark: For a different derivation of the Euler-Lagrange equations, see [1, Appendix B].

References

[1] Vicent Caselles, Ron Kimmel, and Guillermo Sapiro. Geodesic active contours. *International journal of computer vision*, 22(1):61–79, 1997.