

# Variational Methods for Computer Vision: Solution Sheet 9

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Exercise: December 19, 2016

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## Part I: Theory

- Using the property that the delta distribution  $\delta(\phi)$  is the derivative of  $H$ , we get

$$|\nabla H(\phi)| = |\delta(\phi) \cdot \nabla \phi(x)| = \delta(\phi) |\nabla \phi(x)|.$$

So we can rewrite the functional in the following form:

$$\begin{aligned} \mathcal{L} &= f_1 \cdot H(\phi) + f_2(1 - H(\phi)) + \nu |\nabla H(\phi)| \\ &= (f_1 - f_2)H(\phi) + \nu \delta(\phi) |\nabla \phi(x)| + f_2. \end{aligned}$$

So

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \nabla \phi} &= \frac{\partial}{\partial \nabla \phi} \nu |\nabla \phi| \delta(\phi) \\ &= \nu \cdot \delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} &= H'(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi(x)| \\ &= \delta(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi(x)|. \end{aligned}$$

Using both results we can compute

$$\begin{aligned} \frac{\partial E}{\partial \phi} &= \frac{\partial \mathcal{L}}{\partial \phi} - \operatorname{div} \left( \frac{\partial \mathcal{L}}{\partial \nabla \phi} \right) \\ &= \delta(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi(x)| - \nu \operatorname{div} \left( \delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|} \right). \end{aligned}$$

Further simplifying gives

$$\begin{aligned} \operatorname{div} \left( \delta(\phi) \cdot \frac{\nabla \phi}{|\nabla \phi|} \right) &= \frac{d}{dx} \left( \delta(\phi) \cdot \frac{\phi_x}{|\nabla(\phi)|} \right) + \frac{d}{dy} \left( \delta(\phi) \cdot \frac{\phi_y}{|\nabla(\phi)|} \right) \\ &= \left( \frac{d}{dx} \delta(\phi) \right) \cdot \frac{\phi_x}{|\nabla(\phi)|} + \left( \frac{d}{dy} \frac{\phi_x}{|\nabla(\phi)|} \right) \cdot \delta(\phi) \\ &\quad + \left( \frac{d}{dy} \delta(\phi) \right) \cdot \frac{\phi_y}{|\nabla(\phi)|} + \left( \frac{d}{dx} \frac{\phi_y}{|\nabla(\phi)|} \right) \cdot \delta(\phi) \\ &= \delta'(\phi) \cdot \phi_x \cdot \frac{\phi_x}{|\nabla(\phi)|} + \left( \frac{d}{dx} \frac{\phi_x}{|\nabla(\phi)|} \right) \cdot \delta(\phi) \\ &\quad + \delta'(x) \cdot \phi_y \cdot \frac{\phi_y}{|\nabla(\phi)|} + \left( \frac{d}{dy} \frac{\phi_y}{|\nabla(\phi)|} \right) \cdot \delta(\phi) \\ &= \delta(\phi) \operatorname{div} \left( \frac{\nabla \phi}{|\nabla(\phi)|} \right) + \delta'(\phi) \cdot \nabla \phi \cdot \frac{\nabla \phi}{|\nabla(\phi)|} \\ &= \delta'(\phi) |\nabla \phi| + \delta(\phi) \cdot \operatorname{div} \left( \frac{\nabla \phi}{|\nabla(\phi)|} \right). \end{aligned}$$

Finally,

$$\begin{aligned}\frac{\partial E}{\partial \phi} &= \delta(\phi) \cdot (f_1 - f_2) + \nu \delta'(\phi) |\nabla \phi| - \nu \delta'(\phi) |\nabla \phi| - \nu \delta(\phi) \cdot \operatorname{div} \left( \frac{\nabla \phi}{|\nabla(\phi)|} \right) \\ &= \delta(\phi) \left( f_1 - f_2 - \nu \operatorname{div} \left( \frac{\nabla \phi}{|\nabla(\phi)|} \right) \right).\end{aligned}$$

2.

$$\begin{aligned}E(\phi) &= \int g(x) |\nabla H(\phi(x))| dx \\ &= \int g(x) \delta(\phi) |\nabla \phi| dx.\end{aligned}$$

With  $\frac{\partial E}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \operatorname{div} \left( \frac{\partial \mathcal{L}}{\partial \nabla \phi} \right)$  this leads to

$$\frac{\partial \mathcal{L}}{\partial \phi} = \delta'(\phi) g(x) |\nabla \phi|$$

and

$$\frac{\partial \mathcal{L}}{\partial \nabla \phi} = g(x) \delta(\phi) \frac{\nabla \phi}{|\nabla(\phi)|}.$$

Combining both results and further evaluating  $\operatorname{div} \left( g(x) \delta(\phi) \frac{\nabla \phi}{|\nabla(\phi)|} \right)$  gives

$$\begin{aligned}\frac{\partial E}{\partial \phi} &= \frac{\partial L}{\partial \phi} - \operatorname{div} \left( \frac{\partial L}{\partial \nabla \phi} \right) \\ &= \delta'(\phi) g(x) |\nabla \phi| - \operatorname{div} \left( g(x) \delta(\phi) \frac{\nabla \phi}{|\nabla \phi|} \right) \\ &= \delta'(\phi) g(x) |\nabla \phi| - \delta'(\phi) g(x) \frac{\nabla \phi \cdot \nabla \phi}{|\nabla \phi|} - \delta(\phi) \operatorname{div} \left( g(x) \frac{\nabla \phi}{|\nabla \phi|} \right) \\ &= -\delta(\phi) \operatorname{div} \left( g(x) \frac{\nabla \phi}{|\nabla \phi|} \right).\end{aligned}$$